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# STABILITY OF CYLINDRICAL SHELLS

AN ANALYTICAL AND EXPERIMENTAL INVESTIGATION OF THE EFFECTS OF LARGE  
PREBUCKLING DEFORMATIONS ON THE BUCKLING OF CLAMPED THIN-WALLED  
CYLINDRICAL SHELLS SUBJECTED TO AXIAL  
LOADING AND INTERNAL PRESSURE

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by

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Abstract

An analytical and experimental investigation of the effects of large prebuckling deformation on the buckling of thin-walled, clamped, cylindrical shells subjected to combinations of axial loading and internal pressure, has been carried out. These large deformations are caused by edge conditions at the ends of the shells.

Imperfection free test specimens have been provided by the centrifugal casting of a birefringent epoxy resin compound. A carefully executed test program permitted achievement of a one-to-one correspondence between the theoretical and experimental models. The existence of the prebuckling deformations has been demonstrated by means of the photoelastic (photostress) technique. A "two-step" perturbation technique has been used to arrive at the differential equations governing the shell buckling and a solution has been achieved by means of the Galerkin method and application of the IBM 7074 computer.

The role of the nonuniform deformation, in reducing the buckling loads from that predicted by classical linear theory, has been demonstrated by experiment. Good agreement between analysis and experiment has been encountered for shells of limited range of shell lengths.

The inadequacy of the classical membrane model to describe such shells at the incipience of buckling is verified.

Author

### Preface

The objective of this work has been to demonstrate both analytically and experimentally the effect of large nonuniform prebuckling deformations on the buckling of clamped thin-walled, cylindrical shells subjected to combinations of axial compressive loading and internal pressure. These prebuckling deformations arise due to the clamped conditions imposed at the edges of the shells.

The urgent need for such an investigation resulted from recent analytical research work carried out in this field by Stein<sup>[15]</sup> and Fischer.<sup>[16]</sup> They investigated the effects of large prebuckling deformations on the buckling loads of simply supported cylindrical shells. Stein reported reductions of up to 55% from the buckling loads predicted by classical linear theory. Fischer reported reductions of not more than 15%. Koiter<sup>[17]</sup> pointed out that the difference in their findings was probably due in part to the fact that Stein studied the case of vanishing tangential shear at the edges, while Fischer studied the case of vanishing tangential displacement. He also pointed out that Stein's edge conditions did not correspond to those used in the classical linear theory.

It thereby became apparent, that the ultimate answer to the question regarding the role of prebuckling deformations in reducing the buckling loads of thin cylinders would have to be sought in careful experiment. Coupled with this experimental work, an analysis would have to be carried out which provided a one-to-one correspondence with the experiment.

The test specimens have been prepared from a birefringent epoxy resin compound by means of the centrifugal casting technique. Shells have been found to be virtually free of initial geometrical imperfections and the isolation of the effects of the prebuckling deformations in reducing buckling loads from that predicted by classical linear theory has therefore been made possible. Shells have been tested with ratios of radius to thickness varying from 133 to 200, and ratios of length to radius from 0.75 to 4.3. The existence of the prebuckling deformations has been demonstrated by means of the photoelastic (photostress) technique.

The nonlinear Donnell equilibrium equations have been used in the analysis. A solution for the prebuckling problem has been achieved and a "two-step" perturbation technique has been used to arrive at the differential equations governing the shell buckling.

The buckling equations have been solved by means of the Galerkin method and with the aid of an IBM 7074 digital computer.

Results of both the experimental and analytical work have been presented in graphical form and these findings have been discussed at some length.

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# TABLE OF CONTENTS

	PAGE
PREFACE-----	ii
ACKNOWLEDGMENT -----	
LIST OF FIGURES-----	iv
LIST OF SYMBOLS-----	vi
INTRODUCTION-----	1
ANALYTICAL PROCEDURE	
The Equilibrium Equations-----	8
The Prebuckling Deformation-----	11
The Buckling Problem-----	13
EXPERIMENTAL PROCEDURE	
Specimen Preparation-----	27
Testing Apparatus-----	29
Testing Procedure-----	31
PHOTOELASTIC STUDY	
Prebuckling Deformations-----	32
Postbuckled Configurations-----	33
DISCUSSION AND CONCLUSIONS	
Analytical Results	
(a) Computed Buckling Loads-----	34
(b) Effect of Number of Terms in Trigonometric Expansions-----	35
Experimental Results	
(a) Experimental Buckling Loads-----	36
(b) Effect of Shell Length and Ratio of Radius to Thickness-----	37
Comparison of Experimental and Analytical Results-----	37
Conclusions-----	38
APPENDICES	
A. Investigation of Number of Terms Required in Expansions--	63
B. Fortran-Pitt Computer Program (Print-out)-----	65
BIBLIOGRAPHY-----	79

## LIST OF FIGURES

- Fig. 1. Regions Containing Experimental Points Obtained by Various Experimenters.
- Fig. 2. Average Stress vs. Cylinder Shortening as Computed by Von Kármán and Tsien.
- Fig. 3a. Coordinates  $x, y, z$ , and Displacements  $u, v, w$ .
- 3b. Forces and Moments on Element of Wall.
- Fig. 4. Spinning Drum Assembly.
- Fig. 5. Axial Testing Facility for Thin-Walled Cylindrical Shells.
- Fig. 6. Computed Ratio of Maximum Shear Stress, to Maximum Shear Stress with Edge Effects Neglected, for Shell Loaded Axially to 90% of Euler Buckling Load.
- Fig. 7. View of Isochromatics of a Thin Cylindrical Shell Subjected to Axial Loading Equal to 90% of the Classical Buckling Load.
- Fig. 8a. View of Prebuckled Cylindrical Shell Isochromatics.
- 8b. View of Postbuckled Cylindrical Shell Isochromatics.
- 8c. View of Postbuckled Cylindrical Shell  $90^\circ$  Isoclinics.
- Fig. 9. Buckling of an Unpressurized Cylindrical Shell for Various Values of  $J$ . Analysis Based on 12 Term Expansion.
- Fig. 10. Buckling of a Cylindrical Shell under Combinations of Axial Loading and Internal Pressure.  $J$  Varied to Minimize  $P^*$  Throughout.
- Fig. 11. Buckling of an Unpressurized Cylindrical Shell for Various Values of  $J$ .
- Fig. 12. Buckling of a Cylindrical Shell under Combinations of Axial Loading and Internal Pressure. Analysis Based on  $J = 2$  and  $J = 8$ .
- Figs. 13, 14, 15, 16, 17. Buckling of a Cylindrical Shell under Combinations of Axial Loading and Internal Pressure. Analysis Based on  $J = 2$ .
- Fig. 18. Buckling of a Cylindrical Shell under Combinations of Axial Loading and Internal Pressure. (Experimental)

Fig. 19. Buckling of Unpressurized Cylindrical Shells. Analysis Carried out with  $J = 10$ ; 12 Terms in Expansions.

Figs. 20a, 20b. Determinant vs.  $P^*$  for 8 and 12 Term Expansions with  $J = 8$  and  $p = 0$ .

Figs. 21a, 21b. Determinant vs.  $P^*$  for 8 and 12 Term Expansions with  $J = 10$  and  $p = 0$ .

Fig. 22. Experimental Buckling Load vs. Ratio of Length to Radius for an Unpressurized Shell of Fixed Thickness and Radius.

## LIST OF SYMBOLS

D	Flexural rigidity of shell wall = $\frac{Et^3}{12(1-\nu^2)}$
E	Young's modulus
J	Number of peripheral waves around the buckled shell
K	Number of terms employed in trigonometric expansions
L	Shell length
$l$	Shell half-length
$M_x, M_y, M_{xy}$	Resultant bending and twisting moments in shell wall
$N_x, N_y, N_{xy}, Q_x, Q_y$	Shell wall longitudinal and shear stress resultants
p	Internal pressure
P	Axial loading per unit circumference along shell edge
$P_E$	Axial loading per unit circumference along shell edge at classical (Euler) buckling load
$P^*$	$P/P_E$
$p^*$	Ratio of hoop stress due to internal pressure, to axial buckling stress based on classical theory
R	Mean radius of shell
t	Thickness of shell wall
$u^*, v^*, w^*$	Displacements in the x, y, and radial outward directions respectively
$\bar{u}, \bar{w}$	Prebuckling displacements
$u, v, w$	Displacements associated with buckling
$U, V, W$	Functions of "x" used to express the buckling displacements
x, y	Axial and circumferential directions
Z	Shell parameter $\frac{L^2}{Rt} \sqrt{(1-\nu^2)}$



$\nabla^4$	$\frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$
$\epsilon_x, \epsilon_y, \gamma_{xy}$	Longitudinal strains in x and y directions, and shear strain
$\nu$	Poisson's ratio
$\xi$	Distance from edge of shell
$\tau_{\max}$	Maximum shear stress in prebuckled shell wall = $(N_x - N_y)/2t$
$\tau_F$	Maximum shear stress in prebuckled shell neglecting edge effects

The mathematical foundations of the theory of elastic stability were first laid down by Euler. On the basis of this theory Timoshenko<sup>[1]</sup> computed the buckling loads for thin cylindrical shells subjected to axial loading.

Choosing a suitable coordinate system to specify the shell initial configuration, he equated to zero the resultants of the longitudinal, tangential, and radial forces acting on a differential element of the shell. Using Hook's law to relate stresses with strains, and the linearized strain displacement relationships, he arrived at three simultaneous linear homogeneous differential equations of equilibrium relating the shell displacements, end loading, shell structural properties and geometry.

It was assumed that on buckling the shell generators and circumference take on a sinusoidal configuration. The small longitudinal, tangential, and radial displacements were assumed to be of the form (Fig. 3)

$$u^* = A \sin \frac{ny}{R} \cos \frac{m\pi x}{L}$$

$$v^* = B \cos \frac{ny}{R} \sin \frac{m\pi x}{L}$$

$$w^* = C \sin \frac{ny}{R} \sin \frac{m\pi x}{L}$$

with the origin of coordinates taken at one end of the shell. These displacements required that the shell generators divide into  $m$  half waves, and the circumference into  $n$  full waves. Substituting these expressions for the displacements into the differential equations of equilibrium he arrived at a set of three

simultaneous algebraic linear homogeneous equations for the quantities A, B, and C. To compute the buckling load it was required that a non-trivial solution exist for these quantities, i.e., that the determinant of their coefficient matrix be equal to zero. This put a relationship between the shell loading and the integers n, and m. It was then shown that the lowest value of the loading which could satisfy the constraining relationship, with permissible values of m and n was as follows:

$$P_E = \frac{Et^2}{R \sqrt{3(1-\nu^2)}}$$

where  $P_E$  is known as the Euler buckling load per unit circumference along the edge of the shell.

In the case of columns and plates, very good agreement has been found between predictions based on theory and experimental results, however, in the case of thin walled cylindrical shells subjected to axial compressive loading, large discrepancies have been encountered. In experiments carried out by Donnell<sup>[2]</sup> and others<sup>[3]</sup> it was found that these shells buckled under loads of only a fraction of that predicted by theory (Fig. 1).

Donnell<sup>[2]</sup>, and later Donnell and Wan<sup>[4]</sup>, tried to explain this discrepancy on the existence of geometric imperfections as well as residual stresses in the test specimens. Flugge<sup>[5]</sup> attempted to explain it by attributing it to the restraint to radial movement of the shells which was provided by the testing machine or supporting edge plates. Both were able to explain a certain amount of reduction in the critical buckling load, how-

ever, they could not account for the fact that the configuration of the buckled test shell was much different from that predicted by theory.

The reason why thin cylindrical shells do not behave in a manner similar to flat plates when they buckle was investigated by von Kármán and Tsien<sup>[6]</sup>. They showed that on buckling, thin shells can undergo lateral deformations of the order of several wall thicknesses. The relationships which then connect the displacements with the stresses are highly nonlinear. A nonlinear large deflection theory must therefore be used instead of the linear one. The nonlinear theory was laid down by Donnell<sup>[2]</sup>, and von Kármán and Tsien employed it to determine the possible equilibrium configurations of a thin cylindrical shell under axial loading. A deformation form of the type observed in buckled shells was assumed and the Rayleigh-Ritz method was used to obtain a solution.

Von Kármán and Tsien found that there existed other equilibrium configurations in addition to the unbuckled configuration, for loads lower than the Euler critical load (Fig. 2). These other configurations were associated with large deflections in the cylinder walls. While this approach did not indicate that shells must buckle at loads lower than the Euler load it did show that small external disturbances could readily cause shells to "jump" from an unbuckled to a nearby buckled configuration before the Euler load had been reached.

A valuable contribution to the understanding of why experimentally observed large displacement buckling is possible with thin cylindrical shells was made by Yoshimura<sup>[7]</sup>. He used differential geometry to show

that a circular cylindrical shell could be transformed into a set of plane triangles. This transformation required bending of the shell wall. The work of bending of a thin shell wall is relatively small compared to that of membrane compression or extension. This explained the ability of the walls to undergo large displacements due to bending. In contrast, flat plates with edges supported against lateral displacements cannot be deformed to a large deflection buckled configuration without large membrane strains occurring and hence much additional work being supplied by the applied load. The ability of plates to carry increased loads after buckling without undergoing large deformation is thus explained.

In reviewing the extensive literature available on the subject of buckling of thin cylindrical shells it is surprising to find that so little attention has been devoted to investigating the effects of edge conditions on the buckling loads. An explanation, perhaps, may be found in the experimental results obtained by earlier writers.<sup>[2]</sup> In many cases it has been assumed that a shell whose length is greater than three quarters of its diameter may be considered as a shell of infinite length in so far as edge effects are concerned. In other cases the edges are considered to be supported in some way during buckling but prebuckling deformation is either neglected or considered to be uniform throughout and prebuckling bending stresses are assumed to have no effect on the buckling load. These assumptions, convenient though they may be from the viewpoint of the analysis, have been seriously challenged by more recent researchers in this field.

Thielmann<sup>[8]</sup> has criticized the assumption of von Kármán and Tsien, and of later researchers, that buckles are distributed periodically over the entire length of the buckled shell and that shell length has no influence on buckling loads. This assumption has been made in spite of the fact that in experimental tests local buckles are observed. In more recent works by Uemura<sup>[9]</sup>, and Evan-Iwanowski<sup>[10]</sup> the phenomena of localized buckling has been introduced.

In reviewing papers in connection with experimental work carried out by Donnell<sup>[2]</sup> one finds the following statement with regard to edge conditions, "In all the experiments cited in this paper the ends of the cylinders were clamped or fixed in some way. This stabilized the wall of the cylinder near the ends to such an extent that buckling always started at some distance from the ends. When cylinders are tested free ended, eccentricity of loading and other local conditions at the ends are likely to obtain." In a recent report by Tennyson<sup>[11]</sup> it has been claimed that imperfection free cylindrical shells can be made to buckle arbitrarily close to the classical buckling load, with limiting factors being the degree of precision and care used in testing. Leonard<sup>[12]</sup> has completely disagreed with this claim. The following is a quotation from his remarks on the matter, "The author is completely disregarding an important source of error in the classical theory which is entirely unrelated to initial shape imperfections: the inconsistent assumption made in classical theory regarding edge conditions."

A solution to the linearized axisymmetric prebuckling deformation problem for a shell with simply supported edges has been provided by Föppl<sup>[13]</sup> and is presented by Flügge<sup>[4]</sup>. Stein<sup>[15]</sup> computed the solution for the non-linear problem of prebuckling deformations of simply supported cylindrical shells and he computed buckling loads by considering the shell to buckle from this initial nonuniform deformation configuration. He found that the buckling loads were now as little as 45% of those predicted by classical theory.

Fischer<sup>[16]</sup> has investigated a similar problem and has found reduction from the Euler buckling loads, due to prebuckling deformations of about 15%. Koiter<sup>[17]</sup> has pointed out that the differences in Stein's and Fischer's work may be explained in part by the fact that Fischer used the condition of vanishing tangential displacement at the edges while Stein used the condition of vanishing tangential shear. He also stated that since the conditions of Fischer represent those used in the classical membrane problem, a reduction in critical load for Stein's condition of zero tangential shear would likely be obtained even in the case of the membrane solution if Stein's boundary conditions were used. This was shown to be the case by Ohira<sup>[21]</sup> and by Hoff and Rehfield,<sup>[22]</sup>.

Recently, Hoff<sup>[18]</sup> has presented a solution for the axisymmetric buckling of the free end of a thin cylindrical shell. Subsequently Nachbar and Hoff<sup>[19]</sup> have presented a solution to the same type of problem where buckling deformations have not been restricted to the axisymmetric case. In both instances buckling loads well below the Euler loads have been computed.

The objective set forth in this thesis has been to resolve both experimentally and analytically the effects of prebuckling stresses and deformations on the buckling of thin cylindrical shells with clamped edges. It became evident in the early stages of the work that in order to isolate the edge effects experimentally it would be necessary to fabricate test specimens which were virtually free of initial geometric imperfections as well as residual stresses. It furthermore became evident that extreme caution would have to be exercised in fabricating and fitting edge clamping plates, as well as in applying loading to the shells, so that all other possible sources of reduction in buckling loads from the Euler loads would be minimized. In this manner only, could the reduction in buckling loads due to edge effects be determined.

In seeking an analytical solution to this buckling problem it became apparent that the solution must be one which satisfied completely the prescribed experimental boundary conditions. A one-to-one correspondence, thus, between experimental boundary conditions and those formulated mathematically would have to be satisfied. In this manner the reduction in buckling load from the Euler load, due to the effects of clamped edges, would be properly evaluated.



## Analytical Procedure

### The Equilibrium Equations

In order to take into consideration the effects of prebuckling deformations on the buckling of shells, the "two-step" perturbation technique used by Stein<sup>[15]</sup> to arrive at the differential equations governing the buckling is employed here. The large deflection solution for the case of a thin simply supported cylindrical shell subjected to axial and uniform lateral loading, has been provided by Stein<sup>[15]</sup>. The solution for the case of a shell with clamped edge conditions has been computed and is presented here. In both cases the Donnell large deflection equations<sup>[20]</sup> have been used. For completeness a brief review of the development of these equations is presented below.

Referring to Fig. 3 and writing the equations of equilibrium for the forces acting in the x, y, and radial directions respectively we have:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \quad (a)$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \quad (b)$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{N_y}{R} - \left( N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} \right) = p \quad (c) \quad (1)$$

The equilibrium equations for the moments about the x, and y axis respectively are

$$\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = 0 \quad (d)$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \quad (e)$$

Using the relations

$$M_x = \frac{Et^3}{12(1-\nu^2)} \left( \frac{\partial^2 w^*}{\partial x^2} + \nu \frac{\partial^2 w^*}{\partial y^2} \right), \quad M_y = \frac{Et^3}{12(1-\nu^2)} \left( \frac{\partial^2 w^*}{\partial y^2} + \nu \frac{\partial^2 w^*}{\partial x^2} \right)$$

and

$$M_{xy} = \frac{Et^3}{12(1+\nu)} \frac{\partial^2 w^*}{\partial x \partial y}$$

We may write from Eqs. (1)d, (1)e

$$\begin{aligned} \frac{\partial Q_y}{\partial y} + \frac{\partial Q_x}{\partial x} &= \frac{\partial^2 M_y}{\partial y^2} + \frac{\partial^2 M_{xy}}{\partial y \partial x} + \frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{xy}}{\partial y \partial x} \\ &= \frac{Et^3}{12(1-\nu^2)} \left[ \left( \frac{\partial^4 w^*}{\partial y^4} + \nu \frac{\partial^4 w^*}{\partial y^2 \partial x^2} \right) + \left( \frac{\partial^4 w^*}{\partial x^4} + \nu \frac{\partial^4 w^*}{\partial x^2 \partial y^2} \right) + 2 \left( \frac{\partial^4 w^*}{\partial x^2 \partial y^2} - \nu \frac{\partial^4 w^*}{\partial x^2 \partial y^2} \right) \right] \\ &= D \nabla^4 w^* \end{aligned}$$

Substituting in Eq. (1)c, we now have for the set of Donnell equilibrium equations

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \quad (a)$$

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \quad (b) \quad (2)$$

$$D\nabla^4 w^* + \frac{N_y}{R} - (N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2}) = p \quad (c)$$

From Hook's law we have

$$N_x = \frac{Et}{(1-\nu^2)} (\epsilon_x + \nu \epsilon_y) \quad (a)$$

$$N_y = \frac{Et}{(1-\nu^2)} (\epsilon_y + \nu \epsilon_x) \quad (b) \quad (3)$$

$$N_{xy} = \frac{Et}{2(1+\nu)} \gamma_{xy} \quad (c)$$

The nonlinear relationships connecting strains and displacements are,

$$\epsilon_x = \frac{\partial u^*}{\partial x} + \frac{1}{2} \left( \frac{\partial w^*}{\partial x} \right)^2 \quad (a)$$

$$\epsilon_y = \frac{\partial v^*}{\partial y} + \frac{w^*}{R} + \frac{1}{2} \left( \frac{\partial w^*}{\partial y} \right)^2 \quad (b) \quad (4)$$

$$\gamma_{xy} = \frac{\partial u^*}{\partial y} + \frac{\partial v^*}{\partial x} + \frac{\partial w^*}{\partial x} \frac{\partial w^*}{\partial y} \quad (c)$$

### The Prebuckling Deformation

In view of the axial symmetry of the prebuckling problem, it is obvious that both terms on the left hand side of (2)b, are identically zero. Also  $u$  and  $v$  are functions of  $x$  only.

Substituting expressions for stresses in terms of displacements we have from (3)a,

$$N_x = -P = \frac{Et}{(1-\nu^2)} \left[ \frac{du^*}{dx} + \frac{\nu w^*}{R} + \frac{1}{2} \left( \frac{dw^*}{dx} \right)^2 \right] \quad (5)$$

and from (2)c,

$$\frac{d^4 w^*}{dx^4} + \frac{12P}{Et^3} (1-\nu^2) \frac{d^2 w^*}{dx^2} + \frac{12}{R^2 t^2} w^* + \frac{12\nu}{Rt^2} \frac{du^*}{dx} + \frac{6\nu}{Rt^2} \left( \frac{dw^*}{dx} \right)^2 = \frac{12p(1-\nu^2)}{Et^3} \quad (6)$$

Substituting (5) in (6) we obtain

$$\frac{d^4 w^*}{dx^4} + \frac{P^2}{D} \frac{d^2 w^*}{dx^2} + \frac{Et}{R^2 D} w^* = \frac{P}{D} + \frac{P}{DR} \quad (7)$$

The solution to (7) for the case of clamped edges, i.e.,  $w^* = dw^*/dx = 0$  at  $x = \pm L/2$ , has been computed and is as follows:

$$w^* = C_1 \sin \theta x \sinh \phi x + C_2 \cos \theta x \cosh \phi x + q \quad (8)$$

where,

$$q = \frac{R}{Et} \left( p + \frac{\nu P}{R} \right) \quad ; \quad \theta = \frac{1}{2L} \sqrt{4\sqrt{3}z + \frac{PL^2}{D}}$$

$$\phi = \frac{1}{2L} \sqrt{4\sqrt{3}z - \frac{PL^2}{D}}$$

$$C_1 = -2q \frac{\theta \sin \frac{\theta L}{2} \cosh \frac{\phi L}{2} - \phi \cos \frac{\theta L}{2} \sinh \frac{\phi L}{2}}{\theta \sinh \phi L + 2\phi \sin \frac{\theta L}{2} \cos \frac{\theta L}{2}}$$

$$C_2 = -2q \frac{\phi \sin \frac{\theta L}{2} \cosh \frac{\phi L}{2} + \theta \cos \frac{\theta L}{2} \sinh \frac{\phi L}{2}}{\theta \sinh \phi L + 2\phi \sin \frac{\theta L}{2} \cos \frac{\theta L}{2}}$$

differentiating with respect to  $x$ , we obtain

$$\frac{dw^*}{dx} = \gamma_1 \sin \theta x \cosh \phi x + \gamma_2 \cos \theta x \sinh \phi x$$

and

$$\frac{d^2 w^*}{dx^2} = \gamma_3 \sin \theta x \sinh \phi x + \gamma_4 \cos \theta x \cosh \phi x$$

where

$$\gamma_1 = C_1 \phi - C_2 \theta \quad \gamma_2 = C_2 \phi + C_1 \theta$$

$$\gamma_3 = \gamma_1 \phi - \gamma_2 \theta \quad \gamma_4 = \gamma_2 \phi + \gamma_1 \theta$$

The solution for the axisymmetric prebuckled form when  $P$  is greater than  $P_E$  (the Euler loading), that is when the quantity  $\sqrt{4\sqrt{3}z - \frac{PL^2}{D}}$  becomes imaginary, may be expressed as follows

$$w^* = C_1 \sin \theta x \sin \phi x + C_2 \cos \theta x \cos \phi x + q \quad (9)$$

where  $q$  is unchanged but where,

$$C_1 = -q \frac{[\theta \cos \frac{\phi L}{2} \sin \frac{\theta L}{2} + \phi \cos \frac{\theta L}{2} \sin \frac{\phi L}{2}]}{[\theta \sin \frac{\phi L}{2} \cos \frac{\phi L}{2} + \phi \sin \frac{\theta L}{2} \cos \frac{\theta L}{2}]}$$

$$C_2 = -q \frac{[\theta \cos \frac{\theta L}{2} \sin \frac{\phi L}{2} + \phi \sin \frac{\theta L}{2} \cos \frac{\phi L}{2}]}{[\theta \sin \frac{\phi L}{2} \cos \frac{\phi L}{2} + \phi \sin \frac{\theta L}{2} \cos \frac{\theta L}{2}]}$$

and where

$$\theta = \frac{1}{2L} \sqrt{\frac{PL^2}{D} + 4\sqrt{3}z} \quad \phi = \frac{1}{2L} \sqrt{\frac{PL^2}{D} - 4\sqrt{3}z}$$

### The Buckling Problem

Before beginning the calculation of the buckling loads, a consideration of the applicable boundary conditions to be satisfied during buckling is in order. There exist many sets of boundary conditions which are commonly referred to as simply supported or clamped conditions. Four sets of each condition have been discussed in Ref. [24] and are presented here as examples.

Simply support conditions:

- (1)  $w^* = M_{x1} = N_{x1} = v^* = 0$
- (2)  $w^* = M_{x1} = N_{x1} = N_{xy1} = 0$
- (3)  $w^* = M_{x1} = u^* = N_{xy1} = 0$
- (4)  $w^* = M_{x1} = u^* = v^* = 0$

Clamped boundary conditions:

$$(1) \quad w^* = \frac{\partial w^*}{\partial x} = N_{x1} = v^* = 0$$

$$(2) \quad w^* = \frac{\partial w^*}{\partial x} = N_{x1} = N_{xy1} = 0$$

$$(3) \quad w^* = \frac{\partial w^*}{\partial x} = u^* = N_{y1} = 0$$

$$(4) \quad w^* = \frac{\partial w^*}{\partial x} = u^* = v^* = 0$$

Subscripts 1 indicate incremental stress resultants due to buckling. The edge conditions used in buckling tests referred to in this thesis are described by condition (4) in the "clamped boundary conditions" i.e.

$$w^* = \frac{\partial w^*}{\partial x} = u^* = v^* = 0$$

In order to arrive at the differential equations governing the buckling of the shell we add to the prebuckling displacements the infinitesimal buckling displacements  $u$ ,  $v$ , and  $w$ . The total displacements, denoted  $u^*$ ,  $v^*$  and  $w^*$  may thus be written as

$$\begin{aligned} u^* &= \bar{u} + u(x,y) \\ v^* &= v(x,y) \\ w^* &= \bar{w} + w(x,y) \end{aligned} \tag{10}$$

Expressing the three equilibrium equations in terms of these displacements and dropping terms which are products of the infinitesimal buckling displacements  $u$ ,  $v$ , and  $w$ , and making use of expressions involving prebuckling deformations we arrive at the following equilibrium equations<sup>[15]</sup>:

$$\begin{aligned}
& \frac{\partial^2 u}{\partial x^2} + \frac{1-\nu}{2} \frac{\partial^2 u}{\partial y^2} + \frac{(1+\nu)}{2} \frac{\partial^2 v}{\partial x \partial y} + \frac{\nu}{R} \frac{\partial w}{\partial x} + \frac{\partial}{\partial x} \left( \frac{\bar{d}w}{dx} \frac{\partial w}{\partial x} \right) \\
& + \frac{(1-\nu)}{2} \frac{dw}{dx} \frac{\partial^2 w}{\partial y^2} = 0 \\
& \frac{(1+\nu)}{2} \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} + \frac{(1-\nu)}{2} \frac{\partial^2 v}{\partial x^2} + \frac{1}{R} \frac{\partial w}{\partial y} + \frac{(1-\nu)}{2} \frac{d^2 \bar{w}}{dx^2} \frac{\partial w}{\partial y} \\
& + \frac{(1+\nu)}{2} \frac{\bar{d}w}{dx} \frac{\partial^2 w}{\partial x \partial y} = 0 \quad (11)
\end{aligned}$$

$$D\nabla^4 w + \frac{1}{R} N_{yB} + P \frac{\partial^2 w}{\partial x^2} + \nu P \frac{\partial^2 w}{\partial y^2} - \frac{Et}{R} \frac{\bar{w} \partial^2 w}{\partial y^2} - \frac{d^2 \bar{w}}{dx^2} N_{xB} = 0$$

where

$$N_{xB} = \frac{Et}{1-\nu^2} \left[ \frac{\partial u}{\partial x} + \frac{dw}{dx} \frac{\partial w}{\partial x} + \nu \left( \frac{\partial v}{\partial y} + \frac{w}{R} \right) \right]$$

$$N_{yB} = \frac{Et}{1-\nu^2} \left[ \frac{\partial v}{\partial y} + \frac{w}{R} + \nu \left( \frac{\partial u}{\partial x} + \frac{dw}{dx} \frac{\partial w}{\partial x} \right) \right]$$

Admissible expressions for  $u$ ,  $v$ , and  $w$ , in keeping with the requirements of continuity around the cylinder are

$$\begin{aligned}
u &= U(x) \sin \frac{Jy}{R} \\
v &= V(x) \cos \frac{Jy}{R} \\
w &= W(x) \sin \frac{Jy}{R}
\end{aligned} \quad (12)$$



where  $J$  is the number of peripheral waves around the cylinder.  $J$  must be a positive integer greater than 1.  $J = 1$ , represents a translation of the shell and  $J = 0$  represents an axisymmetric form.

Substituting the expressions of (12) into Eqs. (11) we obtain the following set of equations:

$$\begin{aligned}
 \frac{d^2 U}{dx^2} - \frac{1-\nu}{2} \frac{J^2}{R^2} U - \frac{(1+\nu)}{2} \frac{J}{R} \frac{dV}{dx} + \frac{\nu}{R} \frac{dW}{dx} + \frac{d}{dx} \left( \frac{d\bar{w}}{dx} \frac{dW}{dx} \right) \\
 - \frac{(1-\nu)}{2} \frac{J^2}{R^2} \frac{dw}{dx} W = 0 \\
 \frac{(1+\nu)}{2} \frac{J}{R} \frac{dU}{dx} - \frac{J^2}{R^2} V + \frac{1-\nu}{2} \frac{d^2 V}{dx^2} + \frac{J}{R^2} W + \frac{(1-\nu)}{2} \frac{J}{R} \frac{d^2 \bar{w}}{dx^2} W \\
 + \frac{(1+\nu)}{2} \frac{J}{R} \frac{d\bar{w}}{dx} \frac{dW}{dx} = 0 \quad (13) \\
 D \frac{d^4 W}{dx^4} - 2 \frac{DJ^2}{R^2} \frac{d^2 \bar{w}}{dx^2} + \frac{DJ^4}{R^4} W + \frac{1}{R} \overline{N_{yB}} + P \frac{d^2 W}{dx^2} - \nu P \frac{J^2}{R^2} W \\
 + \frac{EtJ^2}{R^3} \frac{\bar{w}}{W} \frac{d^2 \bar{w}}{dx^2} \overline{N_{xB}} = 0
 \end{aligned}$$

where

$$\overline{N_{xB}} = \frac{Et}{1-\nu^2} \left[ \frac{dU}{dx} + \frac{d\bar{w}}{dx} \frac{dW}{dx} + \nu \left( -\frac{J}{R} V + \frac{W}{R} \right) \right]$$

$$\overline{N_{yB}} = \frac{Et}{1-\nu^2} \left[ -\frac{J}{R} V + \frac{W}{R} + \nu \left( \frac{dU}{dx} + \frac{d\bar{w}}{dx} \frac{dW}{dx} \right) \right]$$

A solution for Eqs. (13) in this paper was obtained by means of the Galerkin method. The functions  $U(x)$ ,  $V(x)$ , and  $W(x)$ , appearing in the buckling displacements were expanded in sets of trigonometric functions, each set being selected so that each term of the buckling displacements satisfied completely the prescribed boundary conditions.

It was assumed that buckling is symmetrical about the center of the shell so that only one half of the shell needed to be analysed. The appropriate boundary conditions for the buckling displacements were then as follows:

$$\text{at } x = 0, \quad u = \frac{\partial V}{\partial x} = \frac{\partial w}{\partial x} = \frac{\partial^3 w}{\partial x^3} = 0$$

$$\text{at } x = \frac{L}{2}, \quad u = v = w = \frac{\partial w}{\partial x} = 0$$

In order for these conditions to be fulfilled it was necessary, in view of the choice of expressions for the buckling displacements, that  $U$ ,  $V$ , and  $W$  satisfy the following boundary conditions.

$$\text{at } x = 0, \quad U = \frac{dV}{dx} = \frac{dW}{dx} = \frac{d^3 W}{dx^3} = 0$$

$$\text{at } x = \frac{L}{2}, \quad U = V = W = \frac{dW}{dx} = 0$$

Accordingly, the following expansions were chosen for  $U$ ,  $V$ , and  $W$ .

$$\begin{aligned} U(x) &= \sum_{n=1}^k u_n \sin \frac{n\pi x}{\ell} \\ V(x) &= \sum_{n=1}^k v_n \cos \frac{(2n-1)\pi x}{2\ell} \\ W(x) &= \sum_{n=1}^k w_n \left[ \cos \frac{(n-1)\pi x}{\ell} + \cos \frac{n\pi x}{\ell} \right] \end{aligned} \tag{14}$$

Here  $u_n$ ,  $v_n$ ,  $w_n$  are unknown coefficients. When these expansions for  $U$ ,  $V$ , and  $W$  are substituted into Eqs. (13), each of the three equations contains the  $3k$  unknown coefficients,  $u_n$ ,  $v_n$ ,  $w_n$ . We now multiply each of the three equations by the appropriate trigonometric functions, one at a time, integrate over the interval  $x = 0$ , to  $x = l$ , and setting the result equal to zero (the Galerkin method), thereby obtain  $3k$  linear homogeneous equations for the coefficients.

Finally, we must establish the lowest value of  $P$ , the loading per unit length along the edge of the shell, which will permit their coefficient matrix to have a zero determinant. It is this value of  $P$  which gives the load at which the shell will buckle. Since each choice of  $J$ , the number of peripheral waves around the shell, will have a buckling load associated with it we must investigate different values of  $J$ , to find the lowest of all possible buckling loads.

In presenting the matrix at hand, that is the matrix of the coefficients  $u_n$ ,  $v_n$  and  $w_n$ , denoted herein as matrix  $A$ , it is advantageous at this time to introduce some abbreviations. In addition to employing the Kroniker delta, denoted by the symbol  $\delta$ , the following notation is also used

$$\int_0^l \cosh a_1 x \cos a_2 x \sin a_3 x \sin a_4 x dx =$$

$$= CCSS a_1, a_2, a_3, a_4$$

The first large capital always refers to a hyperbolic function, either  $C$  denoting  $\cosh$  or  $S$  denoting  $\sinh$ . The following three letters represent the trigonometric functions,  $C$  denoting cosine and  $S$  denoting sine, and the following four lower case letters represent quantities appearing as shown above.

Carrying out the integration procedure described earlier, we then obtain for the elements of the matrix A, for

$$0 < n \leq K$$

$$0 < m \leq K$$

$$A_{m,n} = \frac{\ell}{2} \left\{ \frac{(v-1) J^2}{2 R^2} - \frac{(n\pi)^2}{\ell^2} \right\} \delta_{m,n}$$

for

$$K < n \leq 2K$$

$$0 < m \leq K$$

$$A_{m,n} = \frac{(1+v)(2n-1)J}{4R} \left\{ \frac{\sin(2n-1-2m) \pi/2}{(2n-1-2m)} - \frac{\sin(2n-1+2m) \pi/2}{(2n-1+2m)} \right\}$$

for

$$2K < n \leq 3K$$

$$0 < m \leq K$$

$$A_{m,n} = \frac{(1-v)\pi v}{2R} \delta_{m,n-1}$$

$$- \frac{vn\pi}{2R} \delta_{m,n}$$

$$- \gamma_1 \left[ \left( \frac{(n-1)\pi}{\ell} \right)^2 + \frac{(1-v) J^2}{2R^2} \right] \text{CSCS } \phi, \theta, \left\{ \frac{(n-1)\pi}{\ell} \right\}, \left\{ \frac{m\pi}{\ell} \right\}$$

$$- \gamma_2 \left[ \left( \frac{(n-1)\pi}{\ell} \right)^2 + \frac{(1-v) J^2}{2R^2} \right] \text{SCCS } \phi, \theta, \left\{ \frac{(n-1)\pi}{\ell} \right\}, \left\{ \frac{m\pi}{\ell} \right\}$$

$$\begin{aligned}
& - \gamma_1 \left[ \left( \frac{n\pi}{\ell} \right)^2 + \frac{(1-\nu) J^2}{2R^2} \right] \text{CSCS } \phi, \theta, \left\{ \frac{n\pi}{\ell} \right\}, \left\{ \frac{m\pi}{\ell} \right\} \\
& - \gamma_2 \left[ \left( \frac{n\pi}{\ell} \right)^2 + \frac{(1-\nu) J^2}{2R^2} \right] \text{SCCS } \phi, \theta, \left\{ \frac{n\pi}{\ell} \right\}, \left\{ \frac{m\pi}{\ell} \right\} \\
& + \frac{(1-n) \gamma_3 \pi}{\ell} \text{SSSS } \phi, \theta, \left\{ \frac{(n-1) \pi}{\ell} \right\}, \left\{ \frac{m\pi}{\ell} \right\} \\
& + \frac{(1-n) \gamma_4 \pi}{\ell} \text{CCSS } \phi, \theta, \left\{ \frac{(n-1) \pi}{\ell} \right\}, \left\{ \frac{m\pi}{\ell} \right\} \\
& - \frac{n\pi\gamma_3}{\ell} \text{SSSS } \phi, \theta, \left\{ \frac{n\pi}{\ell} \right\}, \left\{ \frac{m\pi}{\ell} \right\} \\
& - \frac{n\pi\gamma_4}{\ell} \text{CCSS } \phi, \theta, \left\{ \frac{n\pi}{\ell} \right\}, \left\{ \frac{m\pi}{\ell} \right\}
\end{aligned}$$

for

$$0 < n \leq K$$

$$K < m \leq 2K$$

$$A_{m,n} = \frac{(1+\nu) Jn}{2R} \left\{ \frac{\sin(n-m+1/2)\pi}{(2n-2m+1)} + \frac{\sin(n+m-1/2)\pi}{(2n+2m-1)} \right\}$$

for

$$K < n \leq 2K$$

$$K < m \leq 2K$$

$$A_{m,n} = \frac{\ell}{2} \left\{ \frac{-J^2}{R^2} - \frac{(1-\nu)(2n-1)^2 \pi^2}{8\ell^2} \right\} \delta_{m,n}$$

for

$$2K < n \leq 3K$$

$$K < m \leq 2K$$

$$\begin{aligned}
 A_{m,n} = & \frac{J\ell}{R^2\pi} \left\{ \frac{\sin(2n-2m-1)\pi/2}{(2n-2m-1)} + \frac{\sin(2n+2m-3)\pi/2}{(2n+2m-3)} \right\} \\
 & + \frac{J\ell}{\pi R^2} \left\{ \frac{\sin(n-m+1/2)\pi}{(2n-2m+1)} + \frac{\sin(n+m-1/2)\pi}{(2n+2m-1)} \right\} \\
 & + \left\{ \frac{(1-\nu)}{2} \frac{J}{R} \gamma_3 \right\} \text{SSCC } \phi, \theta, \left\{ \frac{(n-1)\pi}{\ell} \right\}, \left\{ \frac{(2m-1)\pi}{2\ell} \right\} \\
 & + \frac{(1-\nu)}{2} \frac{J}{R} \gamma_3 \text{SSCC } \phi, \theta, \left\{ \frac{n\pi}{\ell} \right\}, \left\{ \frac{(2m-1)\pi}{2\ell} \right\} \\
 & + \left\{ \frac{(1-\nu)}{2} \frac{J}{R} \gamma_4 \right\} \text{CCCC } \phi, \theta, \left\{ \frac{(n-1)\pi}{\ell} \right\}, \left\{ \frac{(2m-1)\pi}{2\ell} \right\} \\
 & + \frac{(1-\nu)}{2} \frac{J}{R} \gamma_4 \text{CCCC } \phi, \theta, \left\{ \frac{n\pi}{\ell} \right\}, \left\{ \frac{(2m-1)\pi}{2\ell} \right\} \\
 & + \frac{(1+\nu) J (1-n)\pi\gamma_1}{2R\ell} \text{CSSC } \phi, \theta, \left\{ \frac{(n-1)\pi}{\ell} \right\}, \left\{ \frac{(2m-1)\pi}{2\ell} \right\} \\
 & + \frac{(1+\nu) J (1-n)\pi\gamma_2}{2R\ell} \text{SCSC } \phi, \theta, \left\{ \frac{(n-1)\pi}{\ell} \right\}, \left\{ \frac{(2m-1)\pi}{2\ell} \right\} \\
 & - \frac{(1+\nu) J n\pi\gamma_1}{2R\ell} \text{CSSC } \phi, \theta, \left\{ \frac{n\pi}{\ell} \right\}, \left\{ \frac{(2m-1)\pi}{2\ell} \right\} \\
 & - \frac{(1+\nu) J n\pi\gamma_2}{2R\ell} \text{SCSC } \phi, \theta, \left\{ \frac{n\pi}{\ell} \right\}, \left\{ \frac{(2m-1)\pi}{2\ell} \right\}
 \end{aligned}$$

for

$$0 < n \leq K$$

$$2K < m \leq 3K$$

$$\begin{aligned}
 A_{m,n} = & \frac{\sqrt{ET} \, n \, \pi}{2R(1-\nu^2)} \delta_{m,n} \\
 & + \frac{\sqrt{ET} \, n \, \pi}{2R(1-\nu^2)} \delta_{(m-1),n} \\
 & - \frac{ETn\pi\gamma_3}{l(1-\nu^2)} \{SSCC \, \phi, \theta, \{\frac{n\pi}{l}\}, \{\frac{(m-1)\pi}{l}\} + SSCC \, \phi, \theta, \{\frac{n\pi}{l}\}, \{\frac{m\pi}{l}\}\} \\
 & - \frac{ETn\pi\gamma_4}{l(1-\nu^2)} \{CCCC \, \phi, \theta, \{\frac{n\pi}{l}\}, \{\frac{(m-1)\pi}{l}\} + CCCC \, \phi, \theta, \{\frac{n\pi}{l}\}, \{\frac{m\pi}{l}\}\}
 \end{aligned}$$

for

$$K < n \leq 2K$$

$$2K < m \leq 3K$$

$$\begin{aligned}
 A_{m,n} = & \frac{-ETJ \, l}{\pi R^2(1-\nu^2)} \left\{ \frac{\sin(2m-2n-1) \, \pi/2}{(2m-2n-1)} + \frac{\sin(2m+2n-3) \, \pi/2}{(2m+2n-3)} \right. \\
 & \left. + \frac{\sin \pi(m-n+1/2)}{(2m-2n+1)} + \frac{\sin \pi(m+n-1/2)}{(2m+2n-1)} \right\} \\
 & + \frac{ET\nu J\gamma_3}{R(1-\nu^2)} \{SSCC\phi, \theta, \{\frac{(2n-1)\pi}{2l}\}, \{\frac{(m-1)\pi}{l}\} + SSCC\phi, \theta, \{\frac{(2n-1)\pi}{2l}\}, \{\frac{m\pi}{l}\}\} \\
 & + \frac{ET\nu J\gamma_4}{R(1-\nu^2)} \{CCCC\phi, \theta, \{\frac{(2n-1)\pi}{2l}\}, \{\frac{(m-1)\pi}{l}\} + CCCC\phi, \theta, \{\frac{(2n-1)\pi}{2l}\}, \{\frac{m\pi}{l}\}\}
 \end{aligned}$$

for

$$2K < n \leq 3K$$

$$2K < m \leq 3K$$

$$\begin{aligned}
 A_{mn} = & \left\{ D \left( \frac{(n-1)\pi}{\ell} \right)^4 \frac{\ell}{2} + D \left( \frac{n\pi}{\ell} \right)^4 \frac{\ell}{2} + \frac{J^2 D (n-1)^2 \pi^2}{R^2 \ell} \right. \\
 & + \frac{J^2 D n^2 \pi^2}{R^2 \ell} + \frac{Et\ell}{(1-\nu^2)R^2} + \frac{D\ell J^4}{R^4} - \frac{P(n-1)^2 \pi^2}{2\ell} \\
 & \left. - \frac{Pn^2 \pi^2}{2\ell} \right\} \delta_{m,n} \\
 & + \left\{ \frac{D(n-1)^4 \pi^4}{2\ell^3} + \frac{DJ^2(n-1)^2 \pi^2}{R^2 \ell} + \frac{Et\ell}{2R^2(1-\nu^2)} + \frac{D\ell J^4}{2R^4} \right. \\
 & \left. - \frac{P(n-1)^2 \pi^2}{2\ell} + \frac{EtJ^2 q \ell}{2R^3} - \frac{\nu PJ^2 \ell}{2R^2} \right\} \delta_{m,n-1} \\
 & + \left\{ \frac{Dn^4 \pi^4}{2\ell^3} + \frac{DJ^2 n^2 \pi^2}{R^2 \ell} + \frac{Et\ell}{2R^2(1-\nu^2)} + \frac{D\ell J^4}{2R^4} - \frac{Pn^2 \pi^2}{2\ell} \right. \\
 & \left. + \frac{EtJ^2 q \ell}{2R^3} - \frac{\nu PJ^2 \ell}{2R^2} \right\} \delta_{m, n+1}
 \end{aligned}$$



$$- \frac{\nu E T \gamma_1 (n-1) \pi}{R l (1-\nu^2)} \{ \text{CSSC} \phi, \theta, \{ \frac{(n-1) \pi}{l} \}, \{ \frac{(m-1) \pi}{l} \} + \text{CSSC} \phi, \theta, \{ \frac{(n-1) \pi}{l} \}, \{ \frac{m \pi}{l} \} \}$$

$$- \frac{\nu E T \gamma_1 n \pi}{R l (1-\nu^2)} \{ \text{CSSC} \phi, \theta, \{ \frac{n \pi}{l} \}, \{ \frac{(m-1) \pi}{l} \} + \text{CSSC} \phi, \theta, \{ \frac{n \pi}{l} \}, \{ \frac{m \pi}{l} \} \}$$

$$- \frac{\nu E T \gamma_2 (n-1) \pi}{R l (1-\nu^2)} \{ \text{SCSC} \phi, \theta, \{ \frac{(n-1) \pi}{l} \}, \{ \frac{(m-1) \pi}{l} \} + \text{SCSC} \phi, \theta, \{ \frac{(n-1) \pi}{l} \}, \{ \frac{m \pi}{l} \} \}$$

$$- \frac{\nu E T \gamma_2 n \pi}{R (1-\nu^2) l} \{ \text{SCSC} \phi, \theta, \{ \frac{n \pi}{l} \}, \{ \frac{(m-1) \pi}{l} \} + \text{SCSC} \phi, \theta, \{ \frac{n \pi}{l} \}, \{ \frac{m \pi}{l} \} \}$$

$$+ \frac{E T J^2 C_1}{R^3} \{ \text{SSCC} \phi, \theta, \{ \frac{(n-1) \pi}{l} \}, \{ \frac{(m-1) \pi}{l} \} + \text{SSCC} \phi, \theta, \{ \frac{(n-1) \pi}{l} \}, \{ \frac{m \pi}{l} \} \}$$

$$+ \text{SSCC} \phi, \theta, \{ \frac{n \pi}{l} \}, \{ \frac{(m-1) \pi}{l} \} + \text{SSCC} \phi, \theta, \{ \frac{n \pi}{l} \}, \{ \frac{m \pi}{l} \}$$

$$+ \frac{E T J^2 C_2}{R^3} \{ \text{CCCC} \phi, \theta, \{ \frac{(n-1) \pi}{l} \}, \{ \frac{(m-1) \pi}{l} \} + \text{CCCC} \phi, \theta, \{ \frac{(n-1) \pi}{l} \}, \{ \frac{m \pi}{l} \} \}$$

$$+ \text{CCCC} \phi, \theta, \{ \frac{n \pi}{l} \}, \{ \frac{(m-1) \pi}{l} \} + \text{CCCC} \phi, \theta, \{ \frac{n \pi}{l} \}, \{ \frac{m \pi}{l} \}$$

$$- \frac{\nu E T \gamma_3}{R (1-\nu^2)} \{ \text{SSCC} \phi, \theta, \frac{(n-1) \pi}{l}, \frac{(m-1) \pi}{l} + \text{SSCC} \phi, \theta, \frac{(n-1) \pi}{l}, \frac{m \pi}{l} \}$$

$$+ \text{SSCC} \phi, \theta, \{ \frac{n \pi}{l} \}, \{ \frac{(m-1) \pi}{l} \} + \text{SSCC} \phi, \theta, \{ \frac{n \pi}{l} \}, \{ \frac{m \pi}{l} \}$$

$$\begin{aligned}
& - \frac{\nu E \gamma_4}{R(1-\nu^2)} \left\{ \text{CCCC} \phi, \theta, \frac{(n-1)\pi}{l}, \frac{(m-1)\pi}{l} + \text{CCCC} \phi, \theta, \frac{(n-1)\pi}{l}, \frac{m\pi}{l} \right. \\
& \quad \left. + \text{CCCC} \phi, \theta, \frac{n\pi}{l}, \frac{(m-1)\pi}{l} + \text{CCCC} \phi, \theta, \frac{n\pi}{l}, \frac{m\pi}{l} \right\} \\
& + \frac{E \gamma_1 \gamma_3 (n-1)\pi}{4l(1-\nu^2)} \left\{ \text{SCSC} 2\phi, 0, \frac{(n-1)\pi}{l}, \frac{(m-1)\pi}{l} + \text{SCSC} 2\phi, 0, \frac{(n-1)\pi}{l}, \frac{m\pi}{l} \right. \\
& \quad \left. - \text{SCSC} 2\phi, 2\theta, \frac{(n-1)\pi}{l}, \frac{(m-1)\pi}{l} - \text{SCSC} 2\phi, 2\theta, \frac{(n-1)\pi}{l}, \frac{m\pi}{l} \right\} \\
& - \frac{E \gamma_1 \gamma_3 n\pi}{4l(1-\nu^2)} \left\{ \text{SCSC} 2\phi, 2\theta, \frac{n\pi}{l}, \frac{(m-1)\pi}{l} + \text{SCSC} 2\phi, 2\theta, \frac{n\pi}{l}, \frac{m\pi}{l} \right. \\
& \quad \left. - \text{SCSC} 2\phi, 0, \frac{n\pi}{l}, \frac{(m-1)\pi}{l} - \text{SCSC} 2\phi, 0, \frac{n\pi}{l}, \frac{m\pi}{l} \right\} \\
& + \frac{E \gamma_2 \gamma_3 (n-1)\pi}{4l(1-\nu^2)} \left\{ \text{CSSC} 2\phi, 2\theta, \frac{(n-1)\pi}{l}, \frac{(m-1)\pi}{l} + \text{CSSC} 2\phi, 2\theta, \frac{(n-1)\pi}{l}, \frac{m\pi}{l} \right. \\
& \quad \left. - \text{CSSC} 0, 2\theta, \frac{(n-1)\pi}{l}, \frac{(m-1)\pi}{l} - \text{CSSC} 0, 2\theta, \frac{(n-1)\pi}{l}, \frac{m\pi}{l} \right\} \\
& + \frac{E \gamma_2 \gamma_3 n\pi}{4l(1-\nu^2)} \left\{ \text{CSSC} 2\phi, 2\theta, \frac{n\pi}{l}, \frac{(m-1)\pi}{l} + \text{CSSC} 2\phi, 2\theta, \frac{n\pi}{l}, \frac{m\pi}{l} \right. \\
& \quad \left. - \text{CSSC} 0, 2\theta, \frac{n\pi}{l}, \frac{(m-1)\pi}{l} - \text{CSSC} 0, 2\theta, \frac{n\pi}{l}, \frac{m\pi}{l} \right\}
\end{aligned}$$

$$+ \frac{E\gamma_1\gamma_4(n-1)\pi}{4\ell(1-\nu^2)} \left\{ \text{CSSC } 2\phi, 2\theta, \frac{(n-1)\pi}{\ell}, \frac{(m-1)\pi}{\ell} + \text{CSSC } 2\phi, 2\theta, \frac{(n-1)\pi}{\ell}, \frac{m\pi}{\ell} \right.$$

$$\left. + \text{CSSC } 0, 2\theta, \frac{(n-1)\pi}{\ell}, \frac{(m-1)\pi}{\ell} + \text{CSSC } 0, 2\theta, \frac{(n-1)\pi}{\ell}, \frac{m\pi}{\ell} \right\}$$

$$+ \frac{E\gamma_1\gamma_4 n\pi}{4\ell(1-\nu^2)} \left\{ \text{CSSC } 2\phi, 2\theta, \frac{n\pi}{\ell}, \frac{(m-1)\pi}{\ell} + \text{CSSC } 2\phi, 2\theta, \frac{n\pi}{\ell}, \frac{m\pi}{\ell} \right.$$

$$\left. + \text{CSSC } 0, 2\theta, \frac{n\pi}{\ell}, \frac{(m-1)\pi}{\ell} + \text{CSSC } 0, 2\theta, \frac{n\pi}{\ell}, \frac{m\pi}{\ell} \right\}$$

$$+ \frac{E\gamma_2\gamma_4(n-1)\pi}{4\ell(1-\nu^2)} \left\{ \text{SCSC } 2\phi, 0, \frac{(n-1)\pi}{\ell}, \frac{(m-1)\pi}{\ell} + \text{SCSC } 2\phi, 0, \frac{(n-1)\pi}{\ell}, \frac{m\pi}{\ell} \right.$$

$$\left. + \text{SCSC } 2\phi, 2\theta, \frac{(n-1)\pi}{\ell}, \frac{(m-1)\pi}{\ell} + \text{SCSC } 2\phi, 2\theta, \frac{(n-1)\pi}{\ell}, \frac{m\pi}{\ell} \right\}$$

$$+ \frac{E\gamma_2\gamma_4 n\pi}{4\ell(1-\nu^2)} \left\{ \text{SCSC } 2\phi, 0, \frac{n\pi}{\ell}, \frac{(m-1)\pi}{\ell} + \text{SCSC } 2\phi, 0, \frac{n\pi}{\ell}, \frac{m\pi}{\ell} \right.$$

$$\left. + \text{SCSC } 2\phi, 2\theta, \frac{n\pi}{\ell}, \frac{(m-1)\pi}{\ell} + \text{SCSC } 2\phi, 2\theta, \frac{n\pi}{\ell}, \frac{m\pi}{\ell} \right\}$$

## Experimental Procedure

### Specimen Preparation

Thin-walled cylindrical shells were fabricated from an epoxy resin and hardener compound using the centrifugal casting technique. This technique was first discussed in Ref. [11]. The casting facility consisted of an acrylic drum which rotated on a horizontal axis and is shown in Fig. 4. The drum was carefully machined and fitted with close fitting end plates which, in turn, were mounted on brass hubs. A specially selected 1 1/4" dia. ground and hardened steel shaft was passed through these hubs. The shaft was supported at each end by high precision ball bearings located on heavy pedestals. The pedestals were fastened to a concrete base. The assembly was driven by V belt from a 1/2 H.P. variable speed electric drive.

The acrylic drum had inner dimensions of 18" in length and 8" in diameter. The wall was 1/2" thick and the end plates 3/4" thick. Six 250 watt infra-red lamps were used to provide heat and promote curing of the epoxy. The drum was rotated at 1200 RPM during shell curing and it was found to be virtually free of vibration effecting forming of the shells.

In the preparation of a thin cylindrical shell a certain sequence of steps was carried out. These steps are described in order as follows:

- (1) Wipe the inner drum surface with mold release, (Hysol Co. No. AC4-4367 was used.)
- (2) Spin the drum with heat lamps turned on to dry the mold release and heat up the drum.

(3) Cast a shell liner in the drum. This is accomplished by mixing the appropriate amount of Hysol Co. Resin No. R8-2038 with Hysol Hardener No. H2-3404 in proper proportions (100 to 11, Resin to Hardener, by weight) and pouring it into the drum through holes in the end plates. The drum is then rotated for about 3 hours with the heat lamps turned on while the liner hardens. The object of the liner is to remove the effect of any small irregularities that might exist on the inner drum surface. The inner surface of the hardened liner now controls the outer surface of the shell to be cast.

(4) Wipe the inner liner surface with mold release and once again rotate the drum with lamps on to dry the mold release.

(5) Mix the necessary amount of Resin and hardener to provide the required shell thickness and add it to the drum. Rotate the drum, with lamps on, for about 10 hours to completely cure the shell.

(6) Remove the cured shell. This is accomplished by pushing the shell and liner assembly out through one end of the disassembled drum. The liner is then cut free of the shell. The shell is wiped off with trichloroethane and is ready for testing.

The shells produced in the above manner have a number of features which are highly desirable for the purpose of testing. These features may be listed as follows:

(1) Shell geometry is extremely good. Shells produced in this manner, with thicknesses of 0.020 in., 0.025 in. and 0.030 in., were found to have a thickness variation of not more than 0.0005 in. Furthermore, cylindrical shells of various geometry can be readily produced. Since the length and

diameter are determined by those of the drum, almost any dimensions can be achieved by varying drum geometry. Thickness of shell walls is controlled by selecting the proper amount of liquid resin and hardener.

(2) The customary problem of effecting a proper bond at shell wall seams is eliminated since there are no seams.

(3) In view of the method of shell production there are no residual stresses in the walls and no initial deformations.

(4) Given sufficient time between tests (approximately 2 hours) the material of the shells undergoes complete elastic recovery from buckling deformations and they may be (and have been) tested over and over again with the same buckling loads reached in successive tests.

(5) An important additional feature of these shells is the fact that the material from which they are made is translucent and bi-refrangent. A photoelastic analysis of the prebuckling, buckling and postbuckling strains of the shells is thus made possible. The reflective (photostress) technique has been used to study the strains.

Still, and high speed photography have both been used to study the strain distributions. A Budd Co. L.F.Z. large field meter has been employed in all photoelastic studies.

#### Testing Apparatus

Shells were tested in a 4 screw Tinnius Olsen Universal Testing Machine (Fig. 5). Shell end plates were fabricated from 3/4" thick, 10" outer diameter circular steel plates. These plates were ground on both sides. A 1/2" wide, 5/16" deep, concentric circular groove was first machined in each plate. Next

a 3/32" wide, 1/16" deep, circular groove, with outer diameter matching that of the shell was recessed in the center of the first groove. In addition each end plate was fitted with an O-ring seal, while one plate was fitted with a pneumatic fitting, so that pressure or vacuum could be applied to the shell as required.

In preparing a shell for testing the following steps were carried out.

(1) The inner surface was spray painted with reflective aluminum paint. This step was required so that a photoelastic study of the strains could be carried out using the photostress technique. The difficulty encountered in trying to achieve a thin uniform deposit of paint on the surface was overcome with the aid of a small blower. The blower was used to maintain an air stream flowing through the shell. An aluminum spray can was used to maintain a fog of paint in the air stream, the paint being gradually deposited on the shell surface. In this manner a very satisfactory reflective surface was achieved.

(2) One end plate was placed on a level table with the grooved side up. The shell to be tested was then positioned in the groove. Hysol Resin and hardener, mixed as described above, was poured into the groove. Three equally spaced holes of 1/4" diameter which had been drilled into the inner groove allowed the mixture to flow across beneath the shell so that the inner groove and outer groove were each filled up to the level of the upper plate surface. The assembly was then left to cure for about 8 hours. Following the curing the shell was rigidly imbedded in the end plate.

(3) The assembly was then placed in the testing machine and the end plate was fastened with cap screws to the levelling plate (see Fig. 5) which in turn was "spring loaded" against the upper platten of the machine. The end plate for the lower end of the shell was then placed in position on the lower platten and the upper platten was lowered until the shell bottom end entered into the groove of the end plate. The lower groove was then filled with Resin and hardener and left for 8 hours to cure. The shell, which then had rigidly built-in ends, was virtually free of initial stresses at the edge. Now the shell was ready for testing.

#### Testing Procedure

In order to insure that the end plates of the shell remained parallel during testing, a levelling plate was used (see Fig. 5). This plate had 3 levelling screws, threaded through it and resting against the upper platten. The screws were equally spaced on a circle of 11 1/2" diameter. A dial gage was mounted beside each screw in such a way that it indicated changes in distance between the end plates at that point. Initially all dial gages were set to zero. During the testing process the loading was periodically interrupted so that the gage readings could be compared and the levelling screws adjusted as required. In this way parallelism of end plates could be controlled so that the dial gage readings did not differ by more than 0.0005" at buckling.

The loading was also interrupted as required so that photographs of the shell could be taken through the photostress field meter. A 1" x 1" grid, which was traced on the shell outer surface with a grease pencil, made possible the establishment of physical locations of fringe orders and isoclinic lines, etc., observed in these photographs.



### Photoelastic Study

#### Prebuckling Deformations

It is known from the theory of photoelasticity that fringe orders obtained at any point, when conducting isochromatic studies, vary linearly with the maximum shear stress resultant at the point. Using Eqs. (3)b and (4) to express  $N_y$  in terms of displacements we have

$$N_y = \frac{Et}{(1-\nu^2)} \left[ \frac{w}{R} + \nu \left( \frac{du}{dx} + \frac{1}{2} \left( \frac{dw}{dx} \right)^2 \right) \right] \quad (15)$$

Substituting for  $du/dx$  from (5) we obtain

$$N_y = \frac{Et}{(1-\nu^2)} \left[ (1-\nu^2) \frac{w}{R} - (1-\nu^2) \frac{\nu P}{Et} \right] \quad (16)$$

therefore

$$N_y = Et \frac{w}{R} - \nu P \quad (17)$$

In view of the fact that  $N_x$  and  $N_y$  are the principal stresses at any point of the prebuckled shell, the maximum shear stress resultant at any point is given by

$$\frac{N_x - N_y}{2t} = \frac{1}{2t} \left[ P(\nu-1) - \frac{Et w}{R} \right] \quad (18)$$

In fig. 6 the ratio of maximum shear stress resultant to maximum shear stress resultant with edge effects neglected is plotted for a cylindrical shell subjected to a load equal to 90% of the Euler buckling load. Figure 7 is a view of the corresponding isochromatics. These isochromatics are shown in color in Fig.

8a. An interesting and informative study of the agreement between experimental and analytical results is thus made possible. Studies indicate good agreement between theoretical and experimental radial deflections.

The rapid variation in maximum shear stress resultant, predicted by theory and manifested by this succession of rings, is due to the rapid variation in tangential (hoop) stress along the shell. The tangential stress variation, in turn, is due to the rapid variation in radial displacement caused by the clamped condition at the shell edges. Since the radial displacement along the shell is almost of the damped sinusoidal type, the tangential stress variation is rapidly damped out on moving in from the edge of the shell.

It is the existence of these nonuniform stresses and displacements, observed in this photoelastic study, that makes the membrane stress model used in classical linear theory inadequate for describing the actual cylindrical shell at the incipience of buckling. In a correct analysis of buckling behavior the influence of these stresses and deformations must be taken into consideration.

#### Post Buckled Configurations

In almost every test conducted under axial load and without internal pressure the shell buckled into a two tier, six peripheral wave, diamond shape configuration. Photographs of the  $90^\circ$  isoclinics and the isochromatics for a typical shell are shown in Figs. 8b, c. These buckles were located almost midway ( $\pm 1/4''$ ) along the shell. The exact periodicity of the buckles, around the shell, as well as the symmetry observed in the photographs attest to the caution used in fabricating and testing. With internal pressure the number of buckles around the shell increased and the tiers tended to move toward one of the edges.

## Discussion and Conclusions

### Analytical Results

#### (a) Computed Buckling Loads

The analytical computations were carried out on an IBM 7074 digital computer. The print-out of a typical program (Fort-Pitt) is contained in Appendix I. In order to maximize the size of matrices which this computer could handle, the matrices were computed and stored, one section at a time, on a storage tape. Next, with the matrix generating program not required, and having more storage space available in the computer, the determinants of the matrices were evaluated. This largest matrix corresponded to a 24 term expansion of the displacement functions. All the analytical results reported herein are based on a 24 term expansion unless stated otherwise.

In order to conserve computer time the usual custom was to first take a "fast pass" at finding the approximate buckling load. This was done using a 12 term expansion and letting  $P^*$  vary from approximately 0.05 to 1.0 in intervals of 0.05. The buckling load to be predicted was known to be in the neighborhood of the first crossing of the axis (change in sign of the determinant). The next step was to increase the number of expansion terms to the desired level  $\leq 24$ , and investigate the location of the lowest zero using finer increments.

Examining Eq. (8) we note that the quantity  $\theta$ , which determines the wavelength of the trigonometric functions appearing in the prebuckling radial displacements, is independent of shell length. It is therefore to be expected that a proper

analysis pertaining to shells of greater length, and hence more prebuckling waves, will require the use of more terms in the trigonometric expansions and hence larger matrices. The first analysis was therefore carried out on a shell with ratio of length to radius ( $L/R$ ) equal to 0.75. This was the shortest length of shell investigated.

In Fig. 9 the buckling load  $P^*$  vs.  $J$ , the number of peripheral waves, is presented for this shell, based on a 12 term expansion. We note that the load reaches a minimum for  $J = 10$ . In Fig. 10 the computed buckling load vs. internal pressure parameter is given for the same shell, with  $J$  chosen to minimize the load, and number of terms  $K$ , equal to 24.

Analytical results for different shell geometries, with  $J = 2$ , are presented in Figs. 13, 14, 15, and 16. In all cases the buckling loads were found to undergo small increases with pressure at first and then level off and become independent of pressure. In Fig. 17 analytical results are given for an unpressurized shell. Here we note the significant improvement in agreement between experimental and analysis as we increase the number of terms in the expansions from 12 to 24.

(b) Effects of Number of Terms used in Trigonometric Expansions

This investigation was concerned with an unpressurized shell with ratio of length to radius ( $L/R$ ) = 3. Here it was found that the effect of varying  $J$  was much more critical (see Fig. 11). Buckling loads have been computed using 8, 12, and 24 term expansions. It is noted that for  $J = 2$  and  $J = 4$  the values of the predicted buckling loads are equal and sensitivity to the number of terms appears to be nil for  $K > 12$ .

At  $J = 6$  the results become more sensitive to the number of terms and for  $J = 8$ , up to 10, the buckling loads, based on a 24 term expansion, begin to drop quite sharply. At  $J = 12$ , the load begins to increase again, having reached a minimum at  $J = 10$ . In Fig. 12 the buckling load vs. internal pressure parameter is plotted for this shell with values of  $J = 2$  and  $J = 8$ . We note that the load increases rapidly with pressure for  $J = 8$ , and eventually begins to level off at a loading slightly above that obtained for  $J = 2$ . This indicates that the discrepancy between analysis and experiment for longer shells with higher values of  $J$  is centered around the region of zero and low internal pressures only.

On studying the equilibrium equations (Eqs. 13) we note that the parameter  $J$  appears in the first two equations in powers not greater than the second. In the third equation the maximum power to which it appears is the fourth. This means that some components which go into making up the matrix elements associated with the third equation will change by a factor of  $10,000/16$ , as  $J$  changes from 2 to 10. This extreme change brought about by alteration of  $J$  may have a highly significant effect on the number of terms required for proper computation, in particular for longer shells.

A more thorough investigation of the effect of the number of terms on the outcome of such computations is given in Appendix 1.

### Experimental Results

#### (a) Experimental Buckling Loads

Shells with ratios of radius to thickness ( $R/t$ ) ranging from 133 to 200 were tested. The lengths varied from 0.75 to 4.3 radii, the radius in each case being equal to 4 inches.

Experimental buckling loads vs. internal pressure parameter, are presented for various shell geometries in Fig. 10, and Fig. 13 through 18. In all experiments the buckling loads initially increased with internal pressure to about 10% above that of the unpressurized shell. The loads then levelled off and were no longer appreciably effected by increased pressure.

(b) Effects of Shell Length and Ratio of Radius to Thickness

In Fig. 22, the buckling load vs. ratio of length to radius for an unpressurized shell of fixed thickness and radius is presented. The buckling load was found to be almost independent of length for  $L/R \geq 1.5$ . It drops off by about 4% as  $L/R$  decreases to .75. The loads were found not to be appreciably affected by changes in  $R/t$  within the range of geometrics investigated.

Comparison of Analytical and Experimental Results

In figure 10 the analytical and experimental results are presented for a pressurized shell with ratio of length to radius equal to 0.75. The computed and experimental buckling loads for the unpressurized shell agree to within about 2 1/2%. For the internally pressurized shell the experimental and analytically predicted buckling loads both rise with pressure, however, the loads predicted by analysis rise somewhat faster at first. Both curves level out eventually and are no longer effected by pressure.

The analytical and experimental results for a longer unpressurized shell ( $L/R=3.0$ ) with different values of  $J$  used in the analysis are presented in Fig. 11. Here we note that for values of  $J$  from 2 up to 6 we have good agreement between experiment and analysis. For values of  $J$  from 7 up to 10 the experiment and analysis begin to differ quite rapidly, with the disagreement being a maximum at  $J=10$ . For  $J$  greater than 10 the analytical results begin to move up again toward those of experiment. The cause of this disagreement was discussed earlier.

Comparison of experimental and analytical results for pressurized shells of various geometrics is presented in figures 13, 14, 15, and 16. The analysis is restricted here to the case of  $J=2$ . It is noted that the agreement between experiment and analysis is good for zero pressure. While both analytic and experimental buckling loads increase with internal pressure, the increase encountered in experiment is considerably greater than that for the analysis. In both cases the loads level off at higher pressures.

### Conclusions

The effect of nonuniform prebuckling deformations brought about by edge supports, in reducing the buckling loads of clamped thin-walled cylinders subjected to axial and lateral loading, is confirmed by both the experimental and analytical results reported herein.

Reductions from the Euler buckling loads of not more than 15% have been encountered. It is, therefore, apparent that an explanation for the much larger discrepancies more commonly encountered in shell testing will have to be found in the effects of imperfections in the specimens as well as techniques used for supporting of the edges and application of loading.

A study of the governing equations and the analytical results indicates that larger matrices are required for investigating the buckling loads of longer shells ( $L/R > 1.0$ ), in particular where low interval pressures are involved.

It should be pointed out at this time that while the analysis carried out pertains to shells with clamped edges, shells with many other types of edge conditions may be analysed provided that appropriate sets of functions are chosen for expansion of the buckling displacements.

The inadequacy of the classical membrane model to describe the shell in the prebuckling regime has already been discussed. Its inadequacy for describing the shell at buckling is born out by both the experimental and analytical results. The effects of large non-uniform prebuckling deformations must be incorporated into any analysis of the buckling of thin-wall shells subjected to combinations of axial loading and internal pressure.

The isolation of the effects of these deformations has been made possible through the preparation of test specimens which are virtually free of imperfections as well as the high degree of accuracy used in fitting edge supports. The caution used in the application of loading has also been a contributing factor.



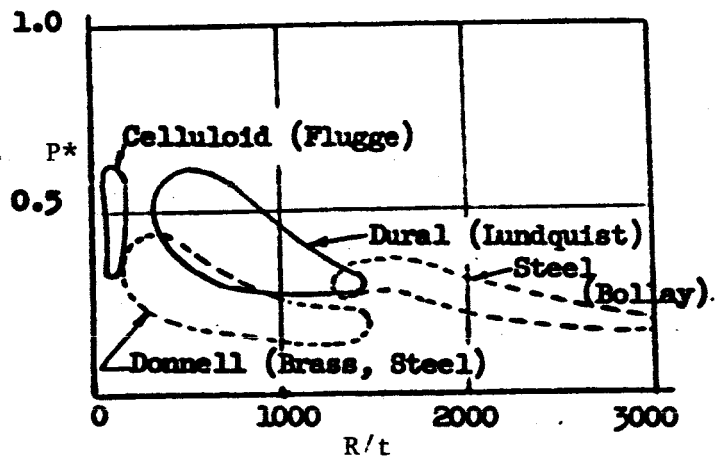


Fig. I. Regions Containing Experimental Points Obtained by Various Experimenters.

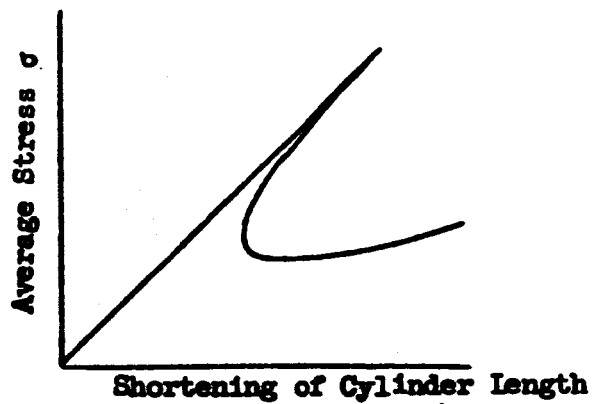


Fig. II. Average Stress vs. Cylinder Shortening as Computed by Von Karman and Tsien.

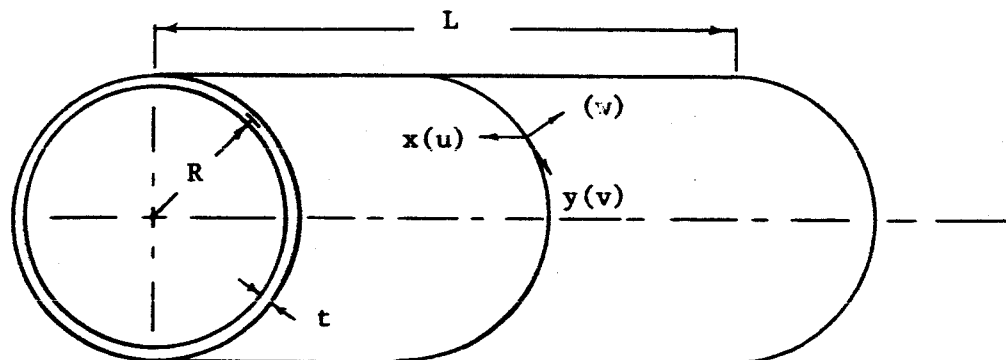


Fig. 3a. Coordinates  $x$ ,  $y$ ,  $z$  and Displacements  $u$ ,  $v$ ,  $w$ .

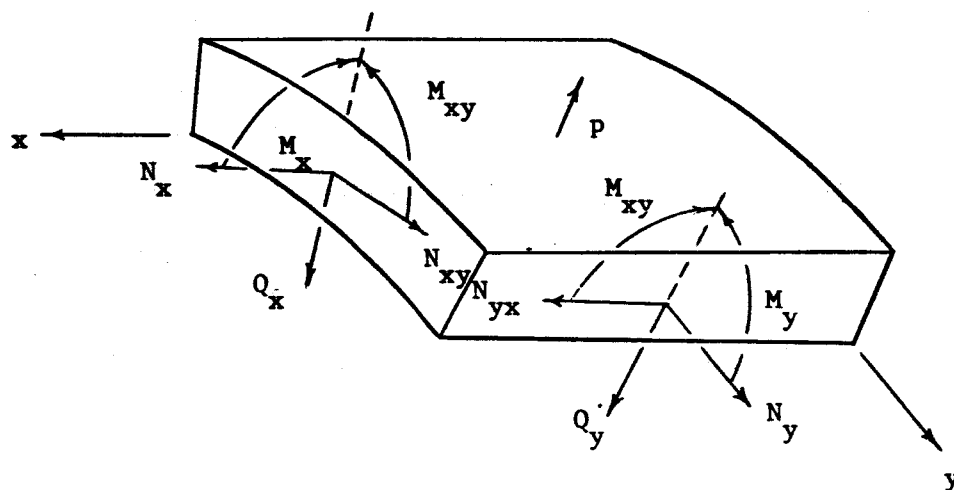


Fig. 3b. Forces and Moments on Element of Wall ( $p$  = Internal Pressure).

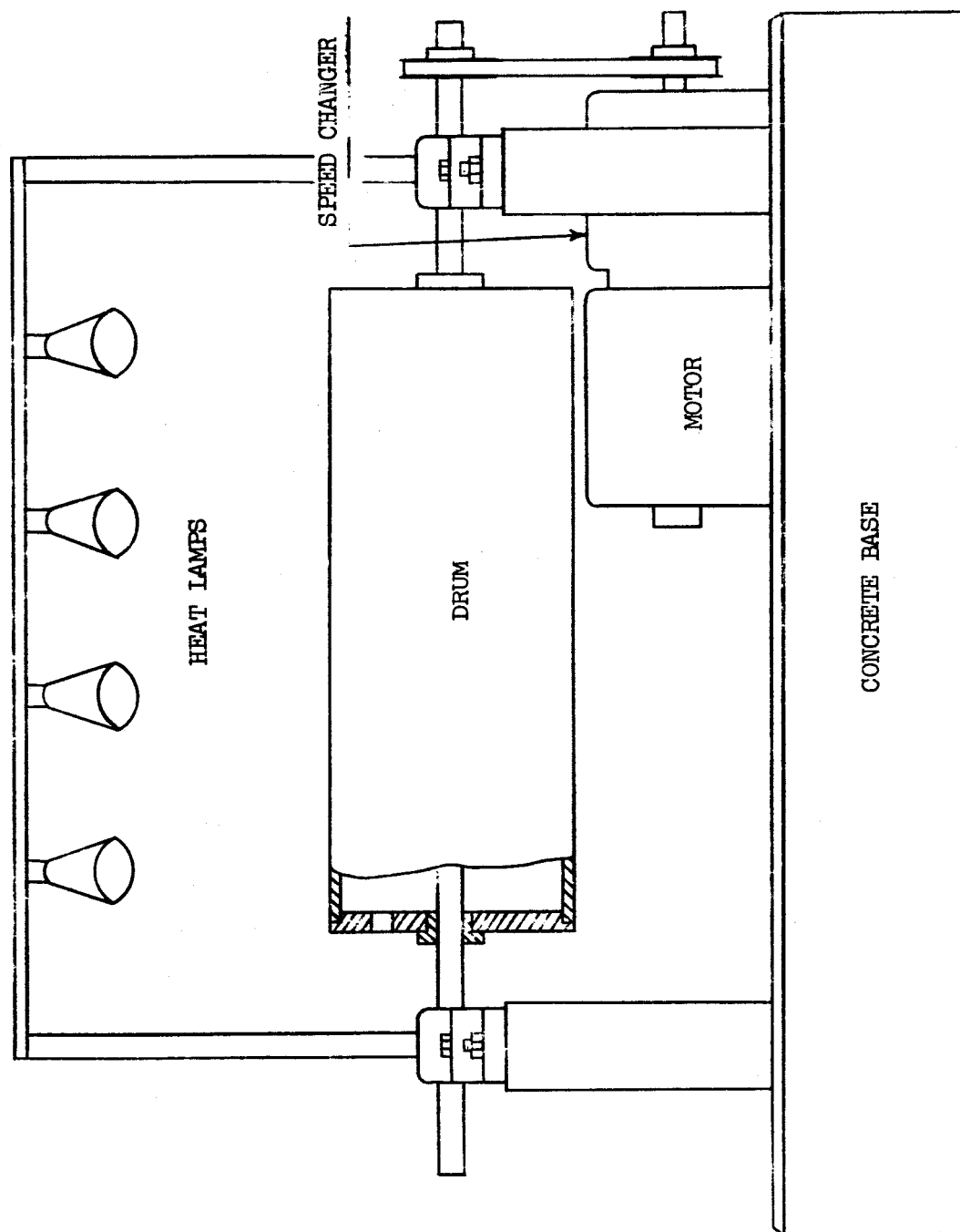


Fig. 4. SPINNING DRUM ASSEMBLY

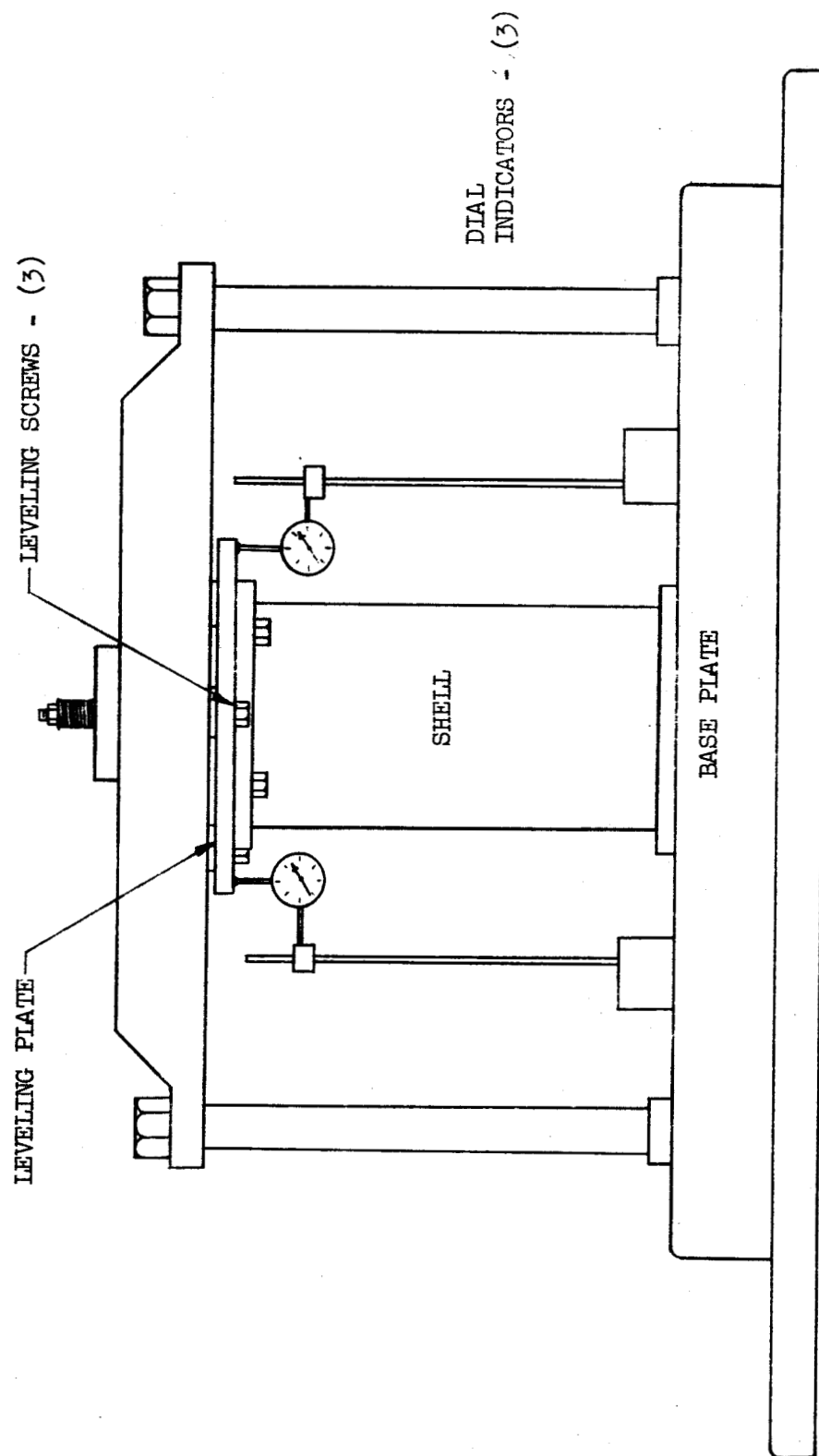


Fig. 5. Axial Testing Facility for Thin Walled Cylindrical Shells.

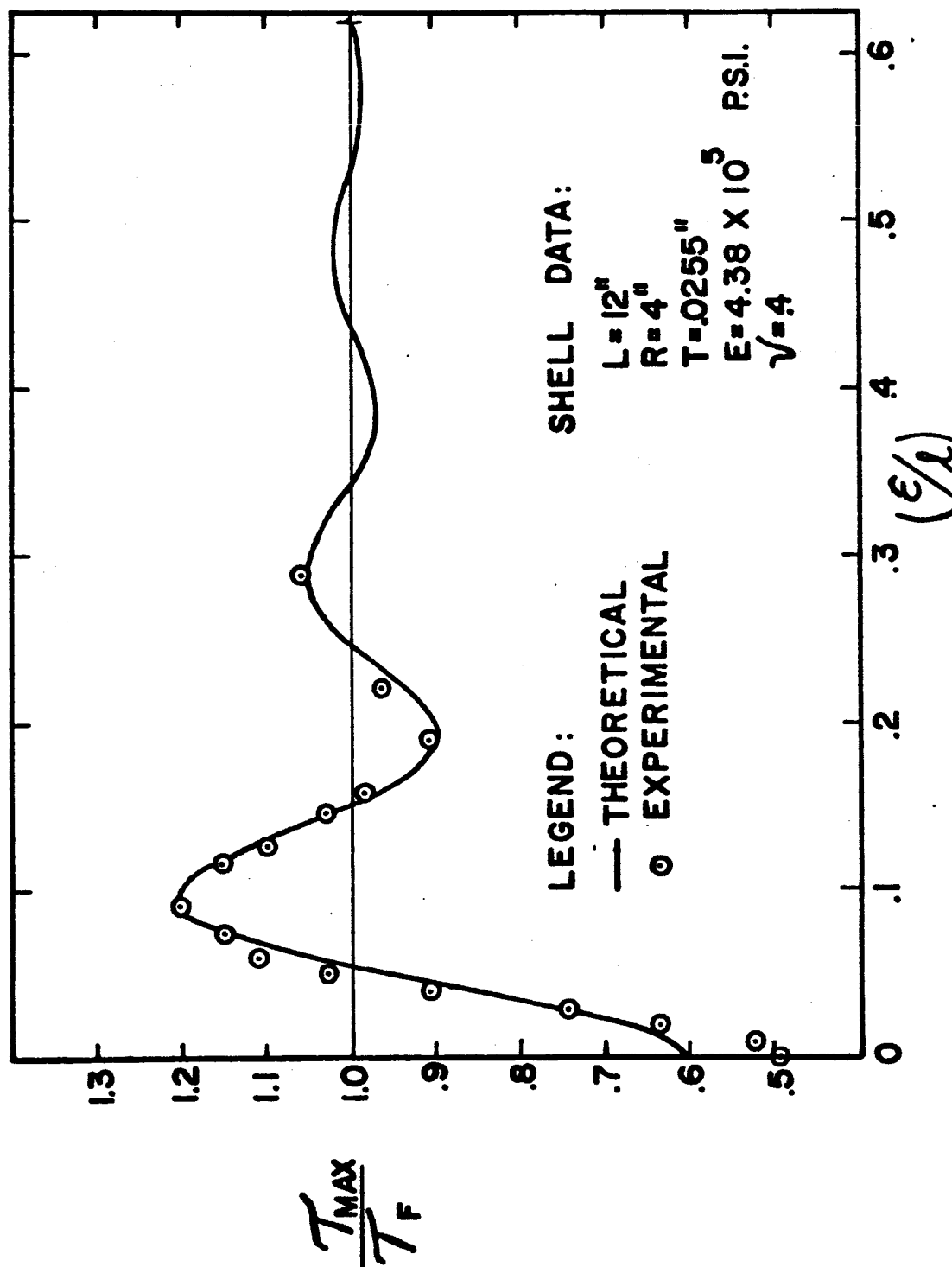


Fig. 6. Computed Ratio of Maximum Shear Stress Resultant to Maximum Shear Stress Resultant with Edge Effects Neglected, for Shell Loaded Axially to 90% of Classical Buckling Load.

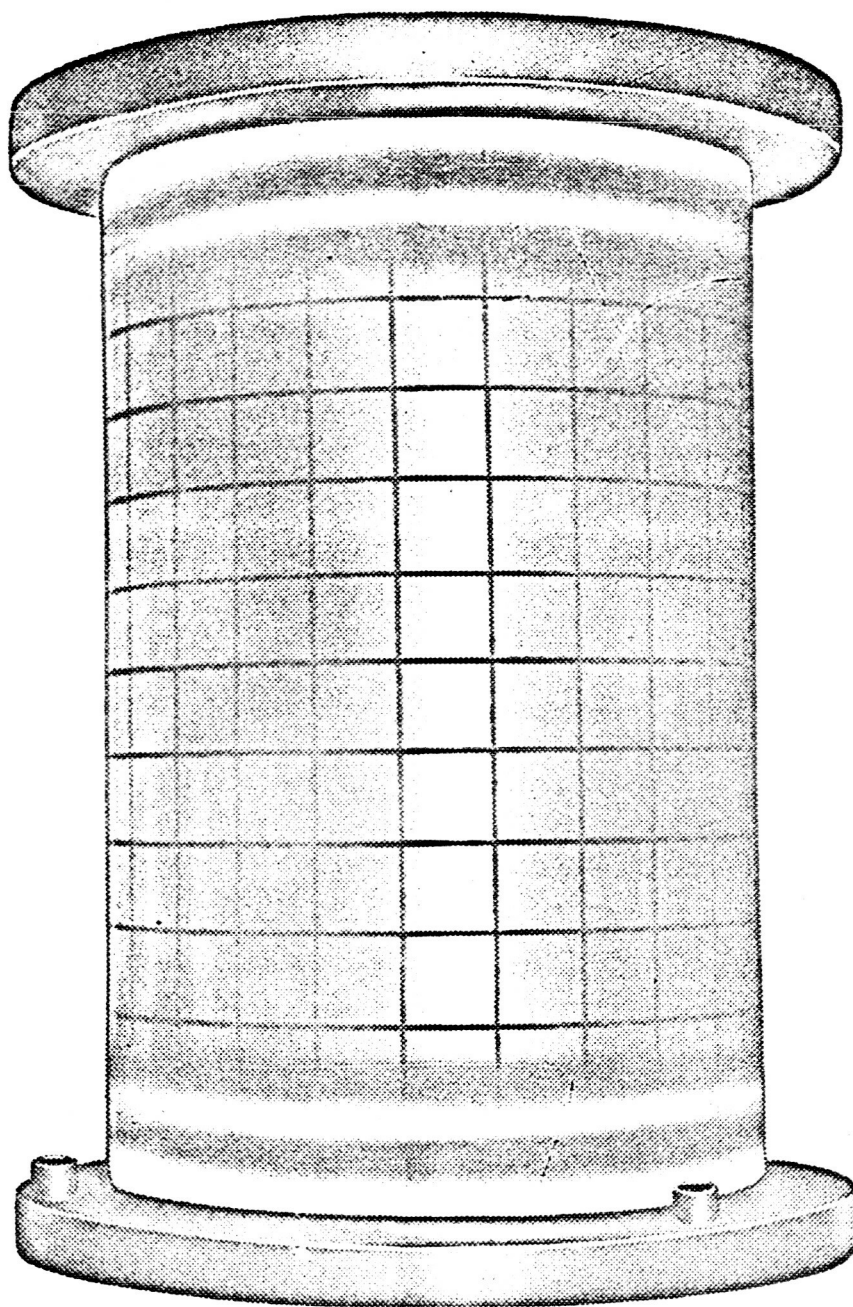


Fig. 7. View of Isochromatics of a Thin Cylindrical Shell Subjected to Axial Loading Equal to 90% of the Classical Buckling Load.

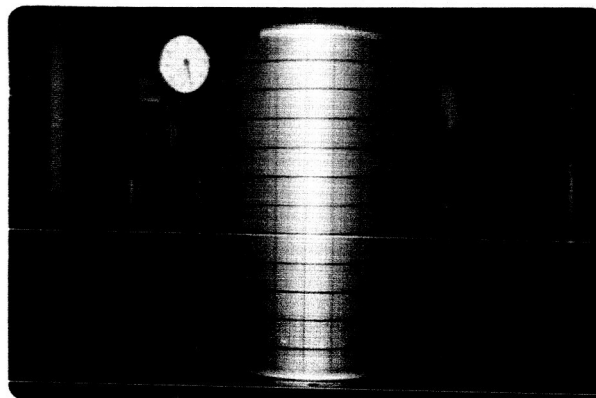


Fig. 8a. View of Prebuckled Cylindrical Shell Isochromatics.

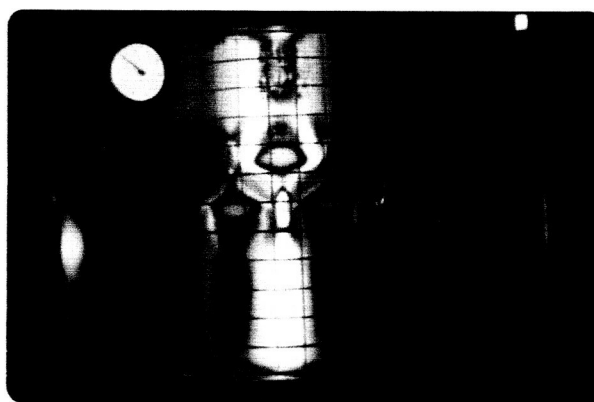


Fig. 8b. View of Post-buckled Cylindrical Shell Isochromatics.

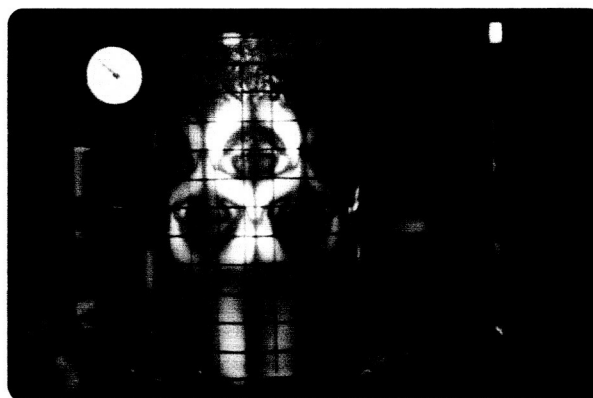


Fig. 8c. View of Post-buckled Cylindrical Shell  $90^\circ$  Isoclinics.

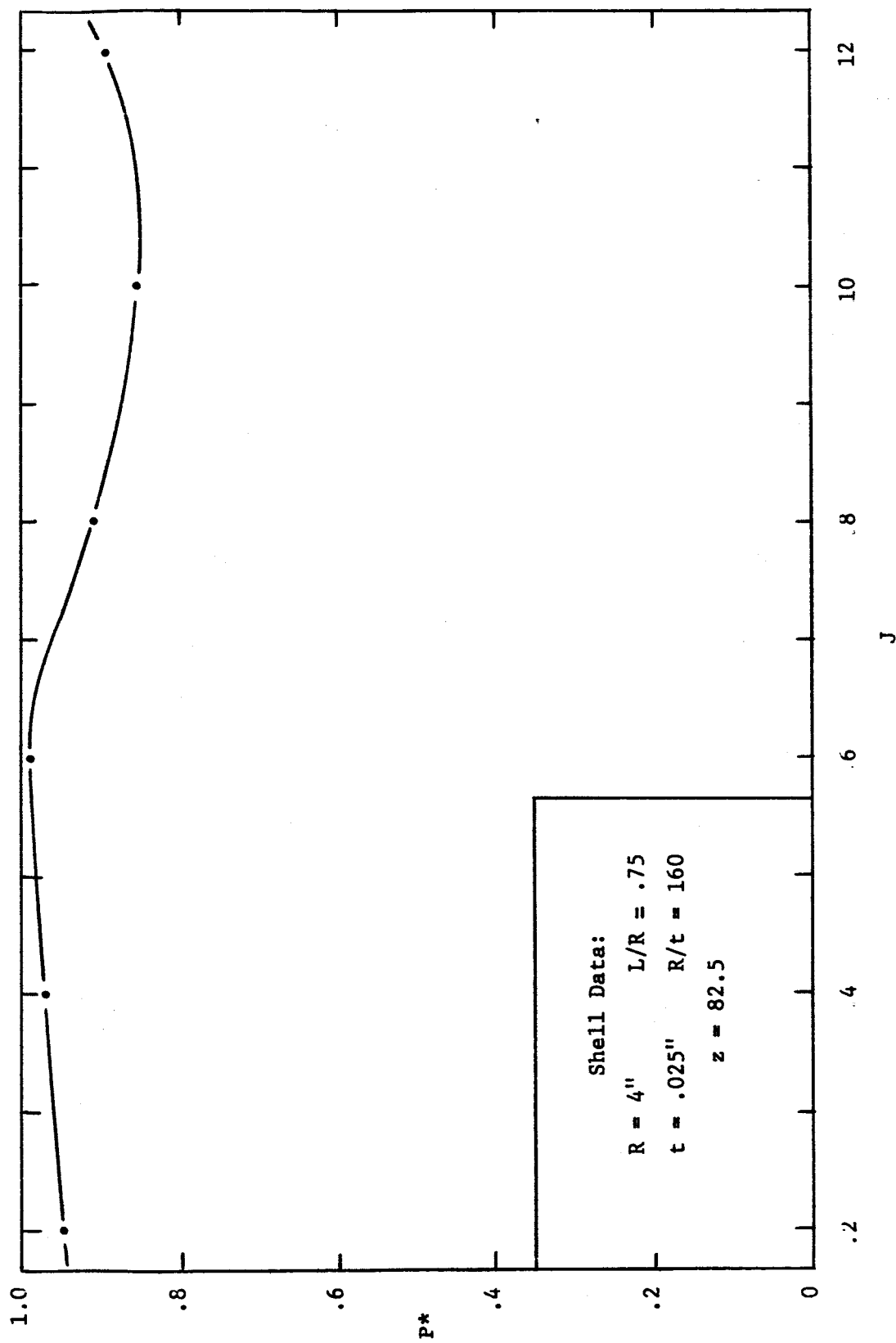


Fig. 9. Buckling of an Unpressurized Cylindrical Shell for Various Values of  $J$ . Analysis Based on 12 Term Expansion.



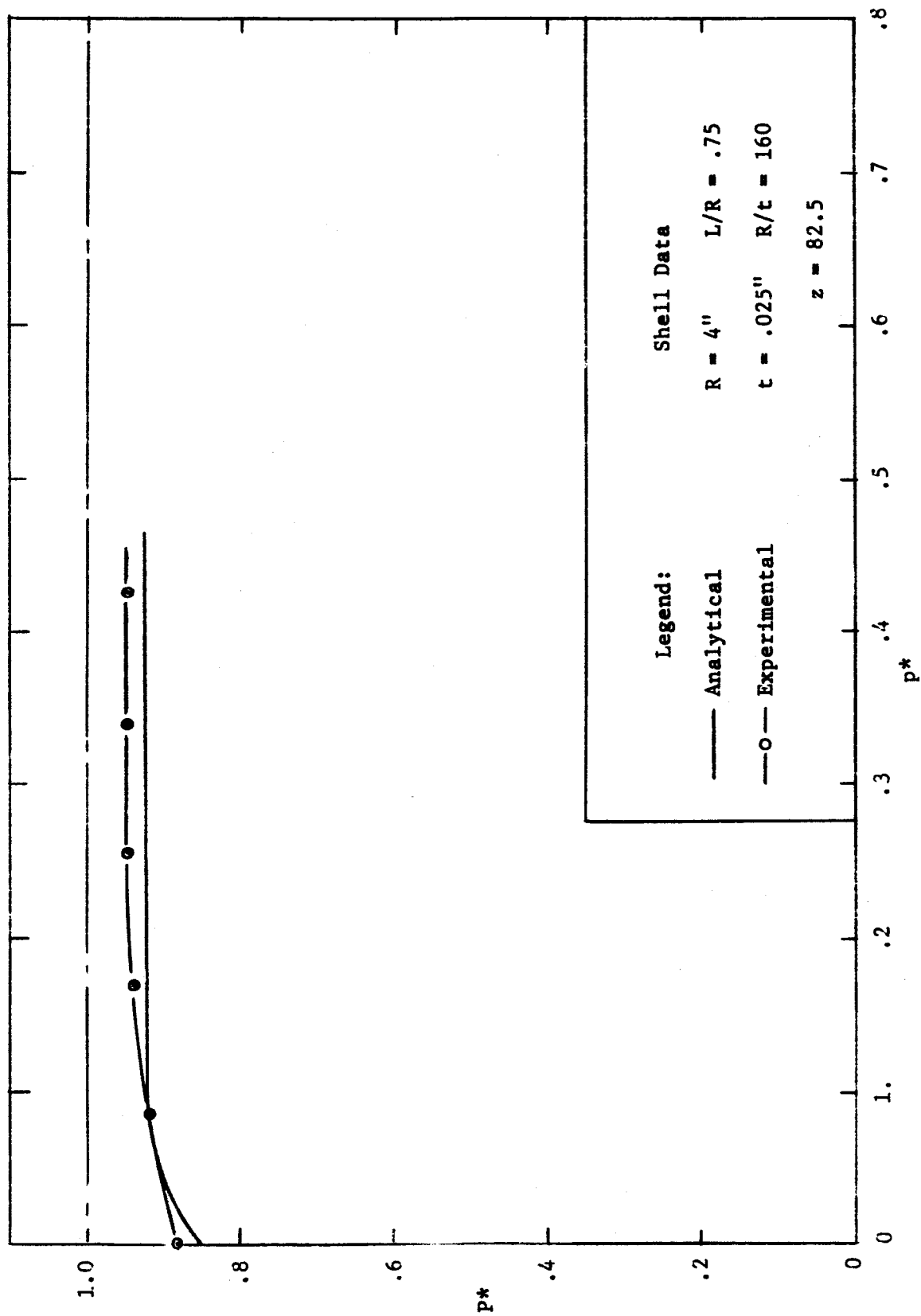


Fig. 10. Buckling of a Cylindrical Shell under Combinations of Axial Loading and Internal Pressure.  
J Varied to Minimize  $p^*$  Throughout.

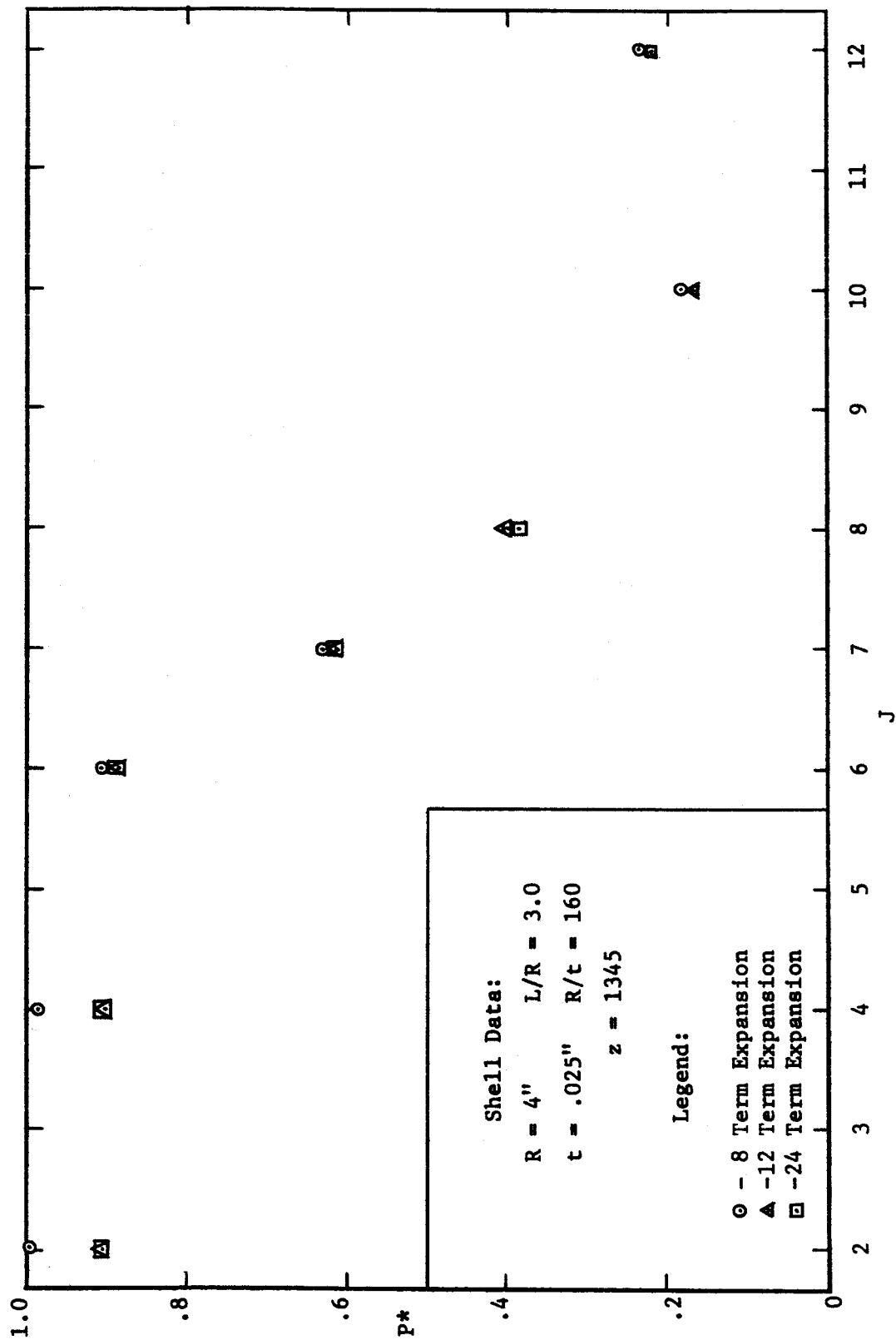


Fig. 11. Buckling of an Unpressurized Cylindrical Shell for Various Values of J.

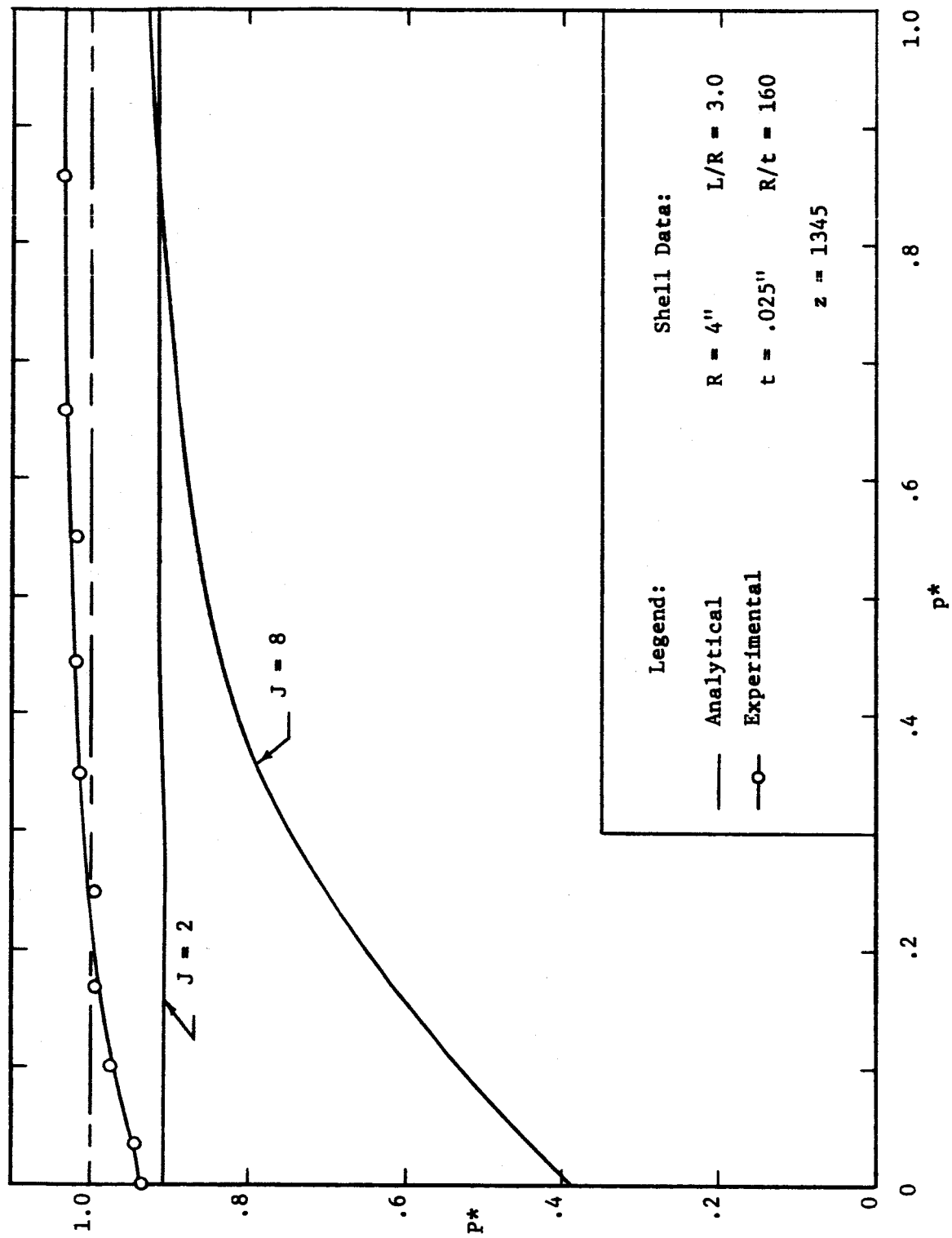


Fig. 12. Buckling of a Cylindrical Shell under Combinations of Axial Loading and Internal Pressure. Analysis Based on  $J = 2$  and  $J = 8$ .

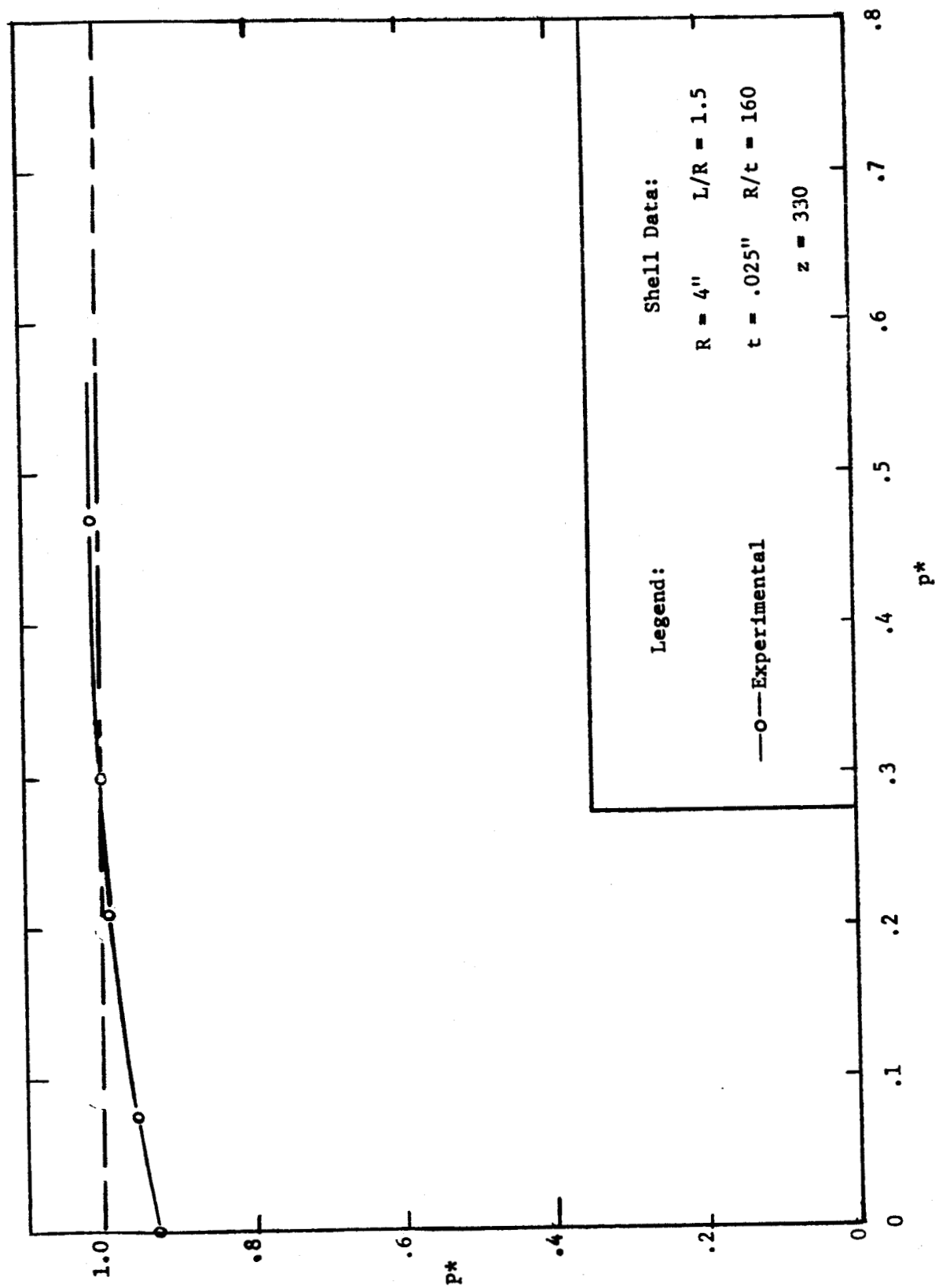


Fig. 13. Buckling of a Cylindrical Shell under Combinations of Axial Loading and Internal Pressure.

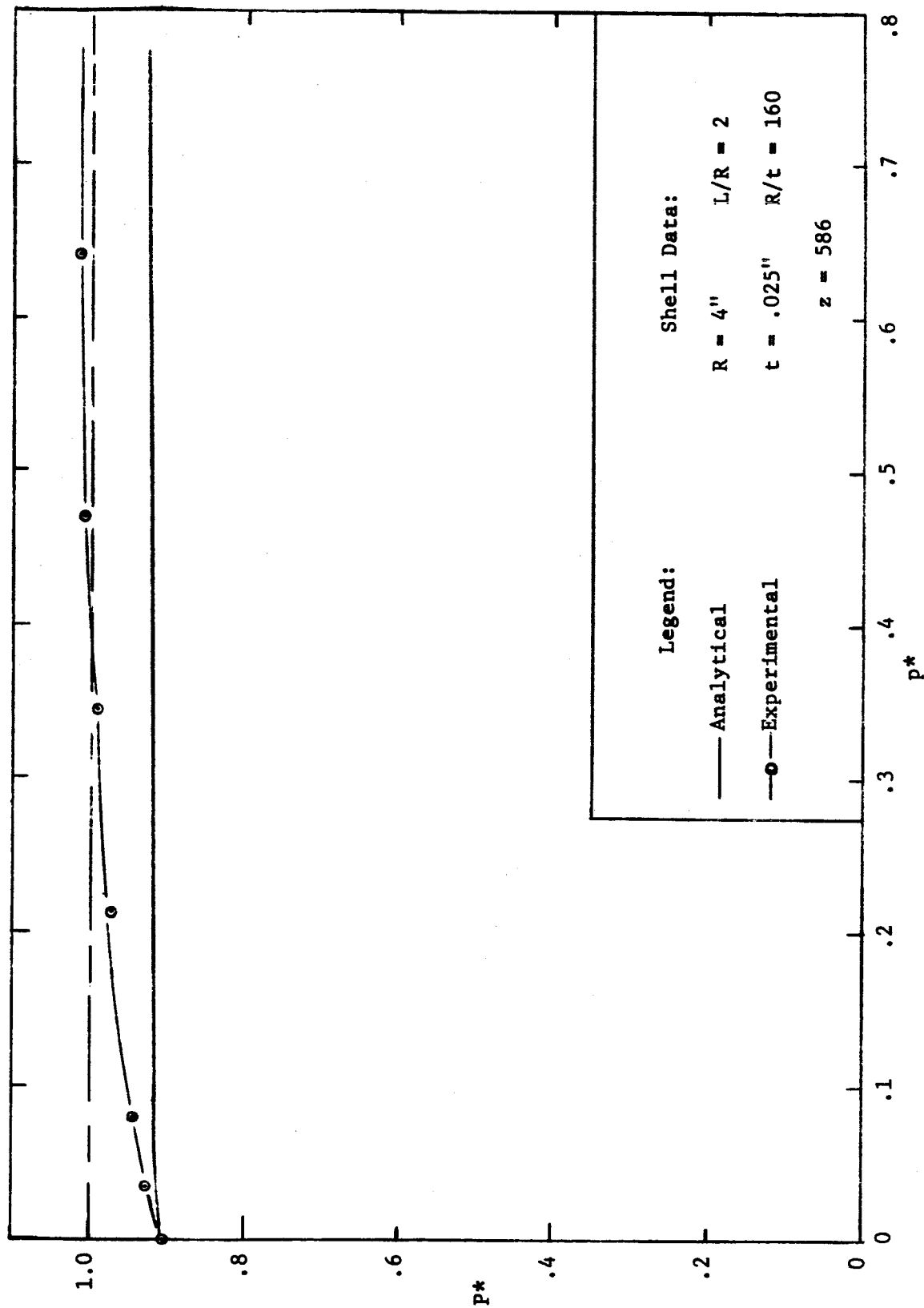


Fig. 14. Buckling of a Cylindrical Shell under Combinations of Axial Loading and Internal Pressure. Analysis Based on  $J = 2$ .

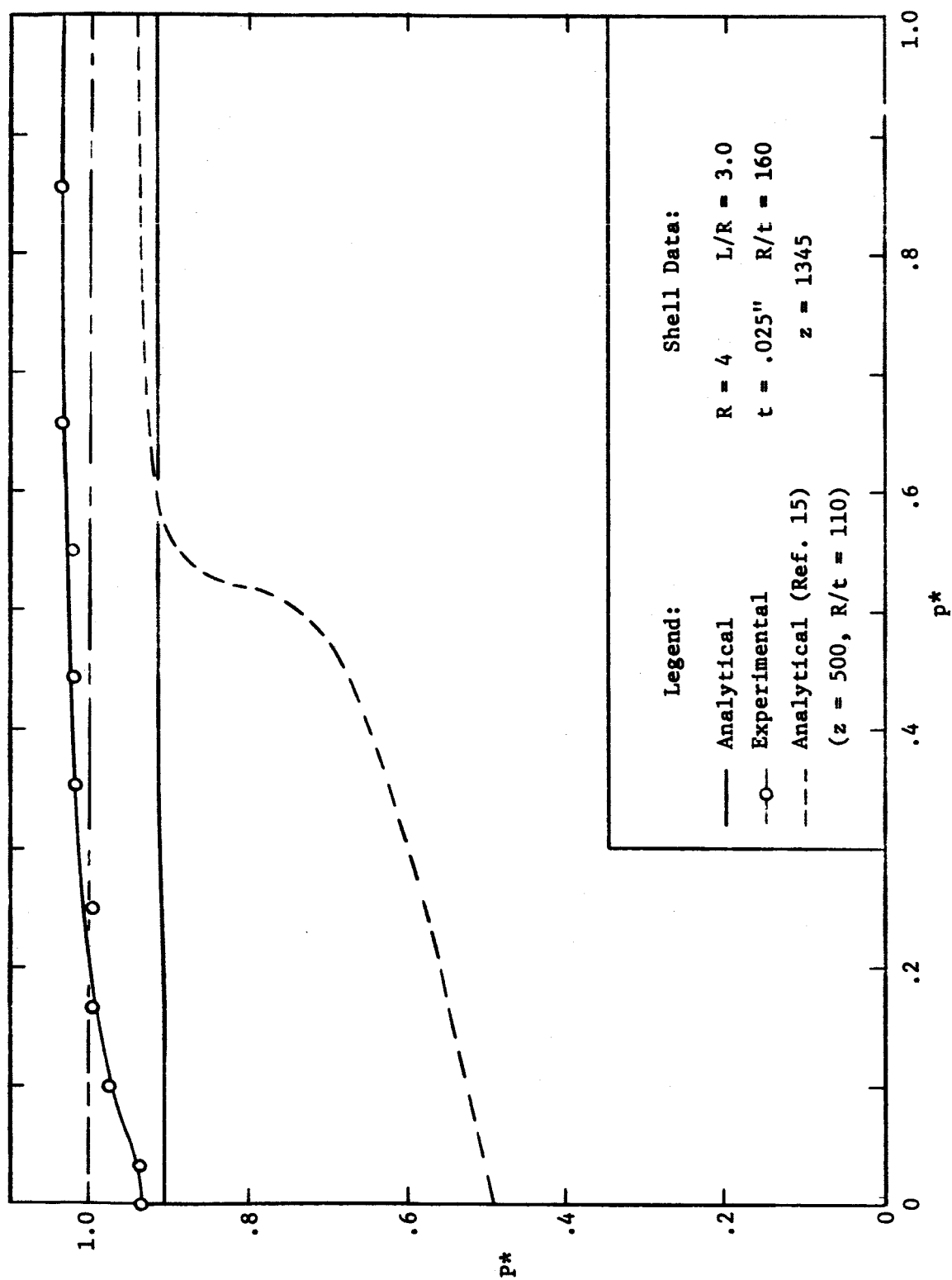


Fig. 15. Buckling of a Cylindrical Shell under Combinations of Axial Loading and Internal Pressure. Analysis Based on  $J = 2$ .

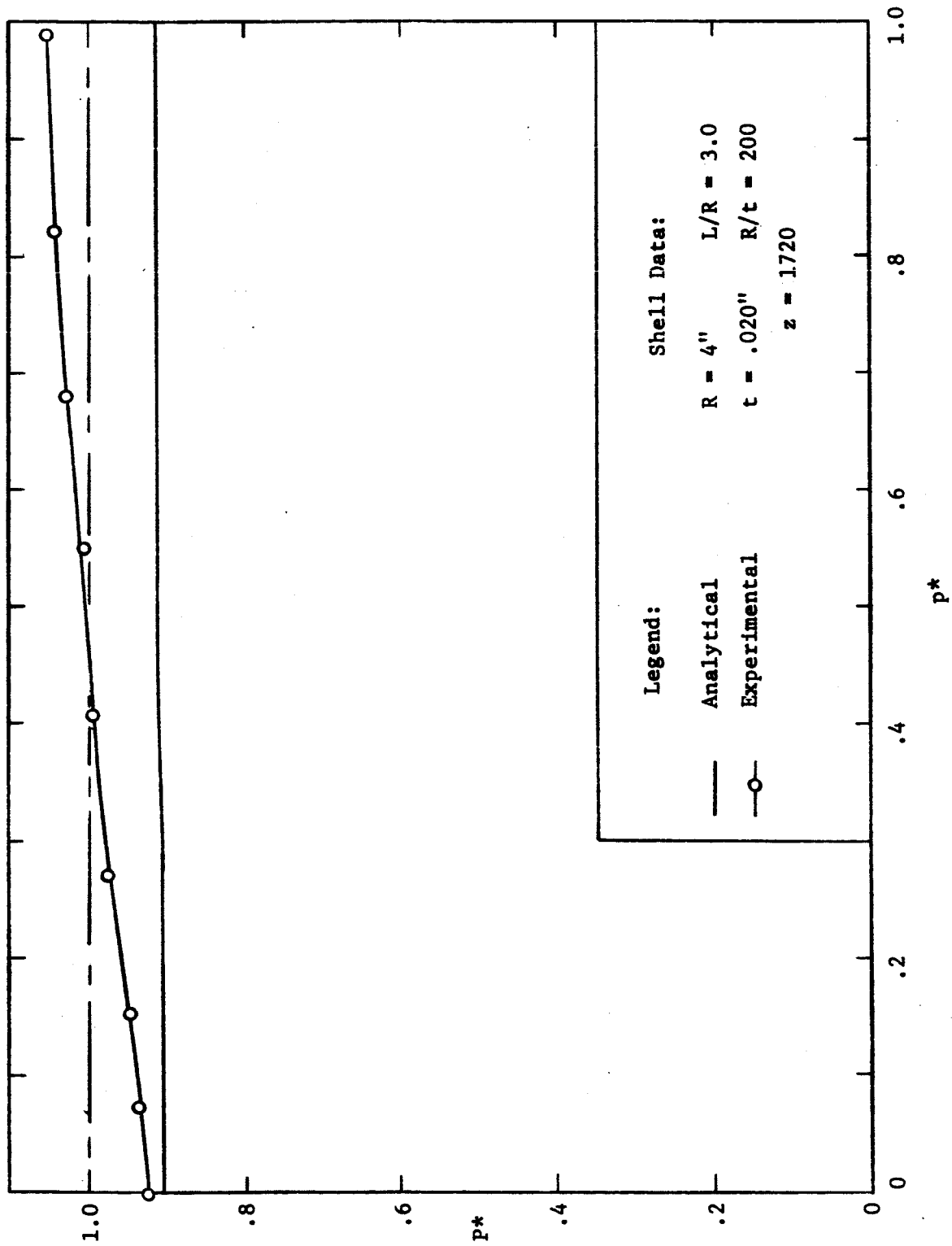


Fig. 16. Buckling of a Cylindrical Shell under Combinations of Axial Loading and Internal Pressure. Analysis Based on  $J = 2$ .

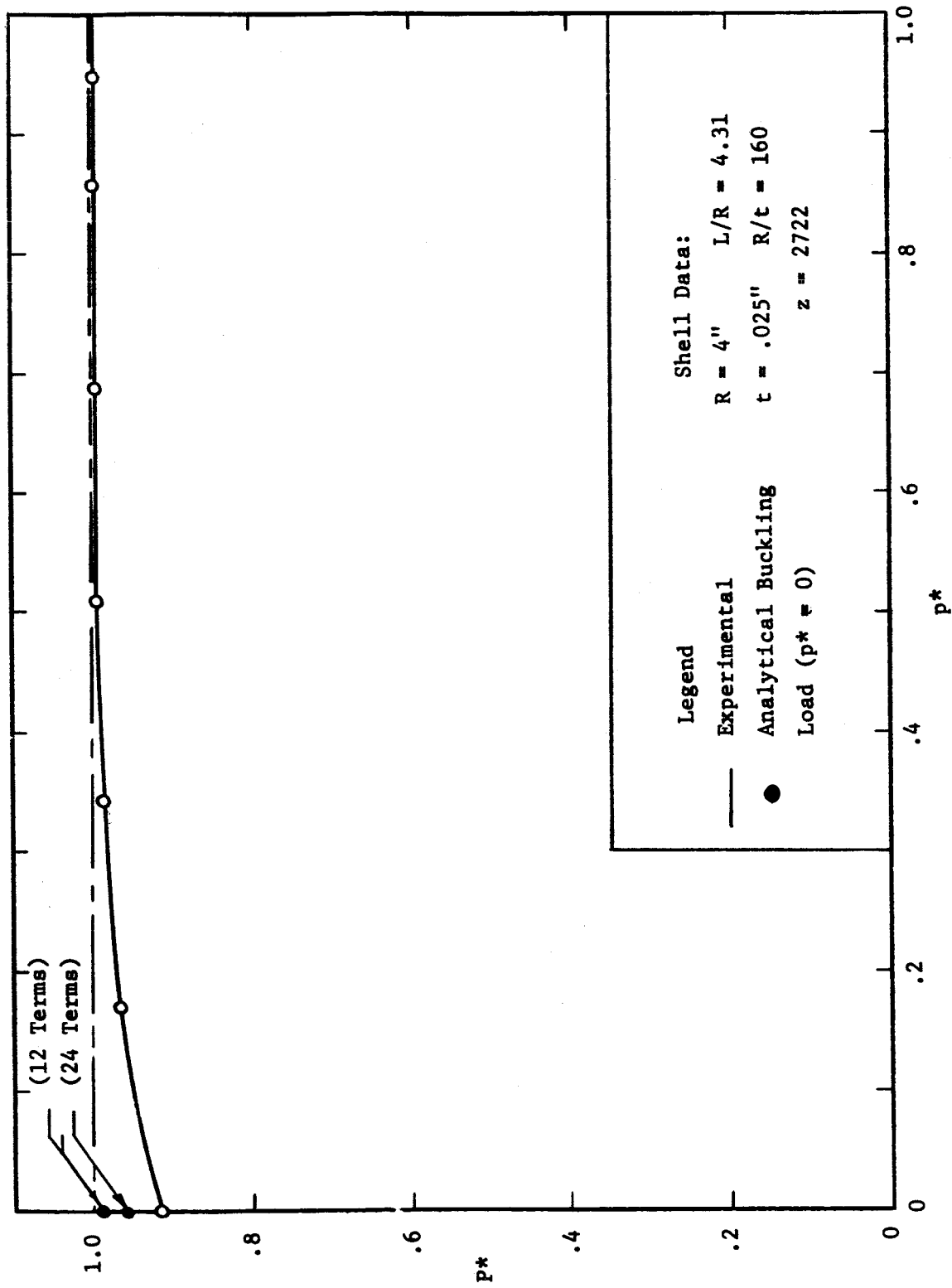


Fig. 17. Buckling of a Cylindrical Shell under Combinations of Axial Loading and Internal Pressure. Analysis Based on  $J = 2$ .



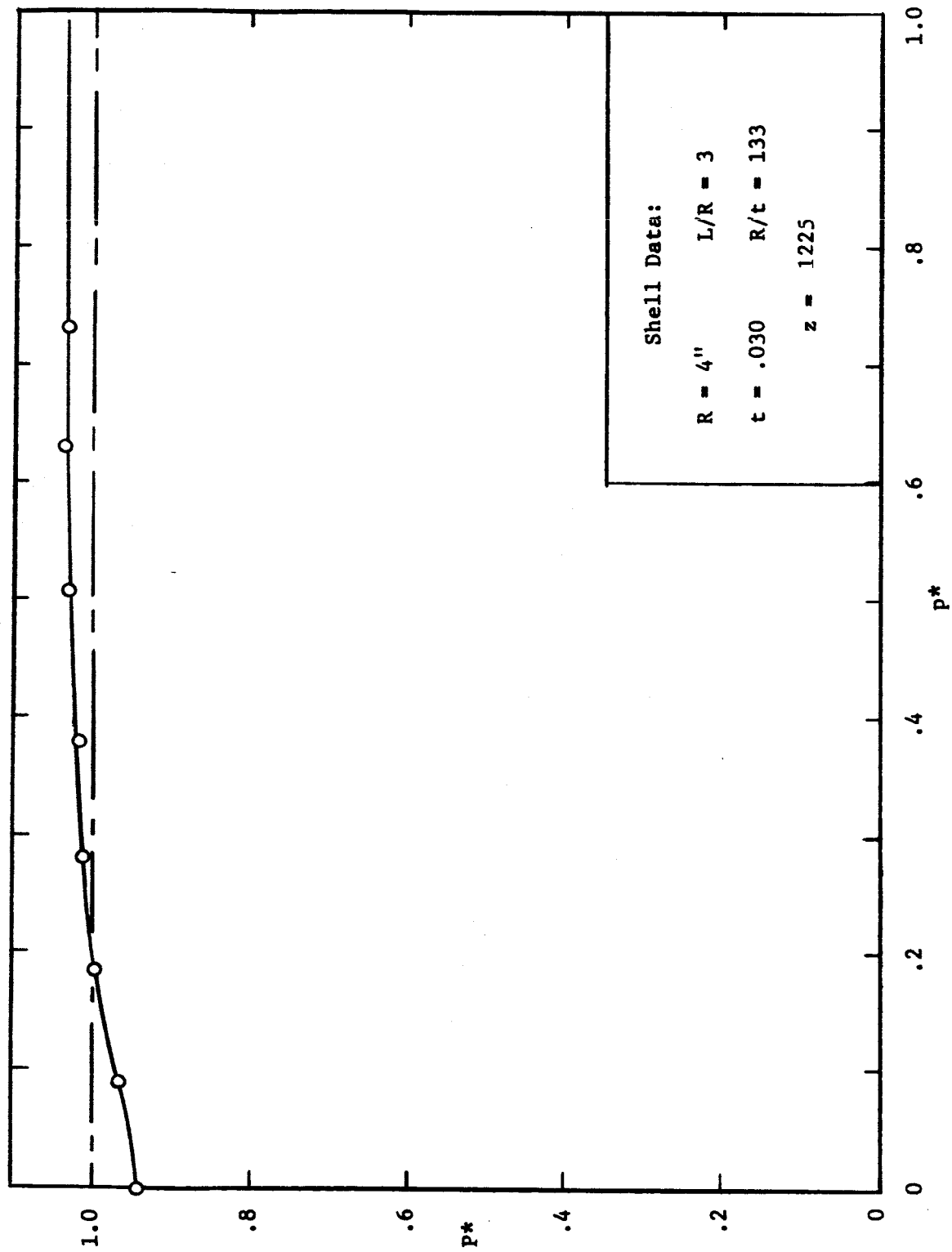


Fig. 18. Buckling of a Cylindrical Shell under Combinations of Axial Loading and Internal Pressure. (Experimental)

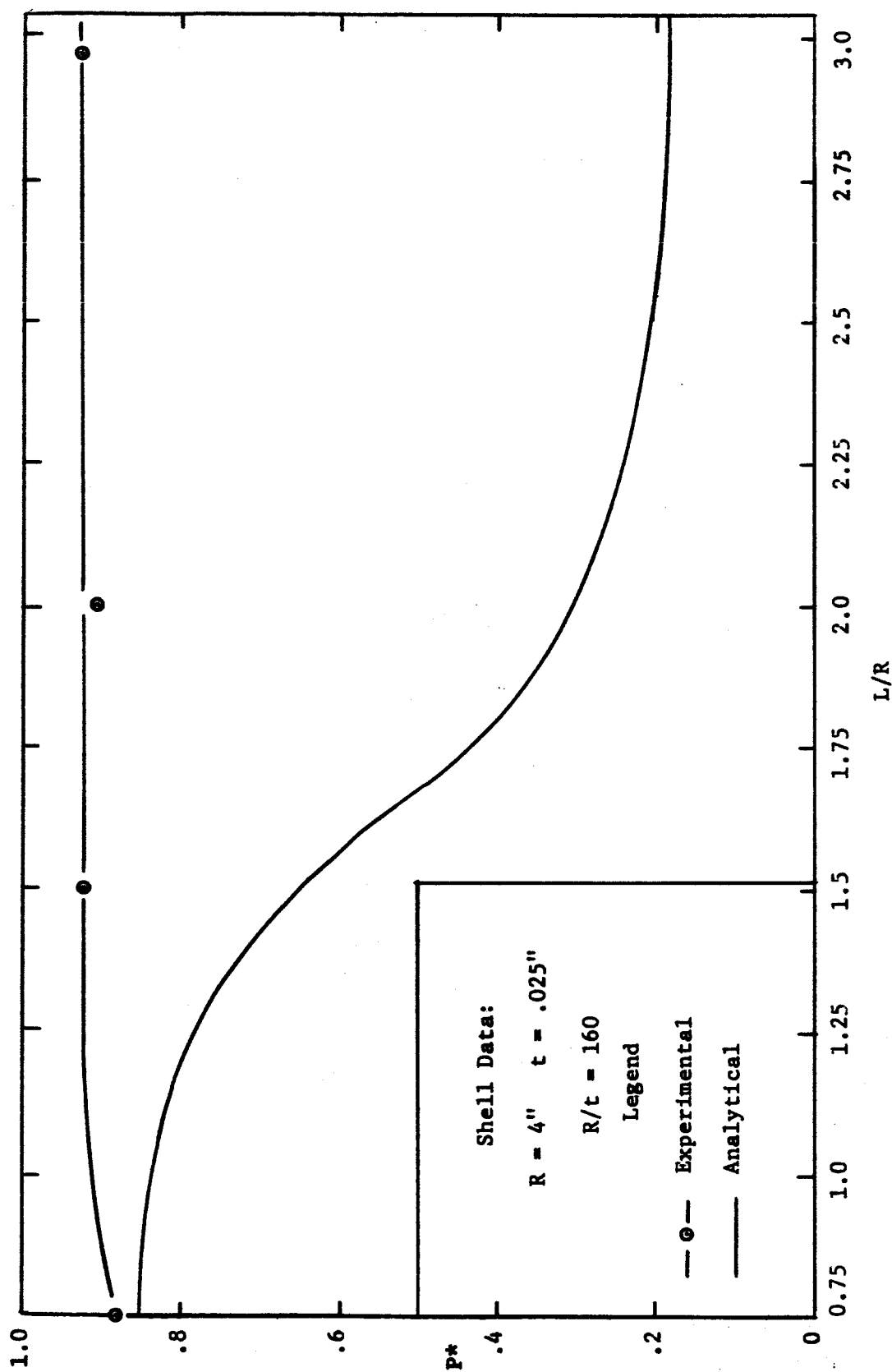


Fig. 19. Buckling of Unpressurized Cylindrical Shells. Analysis Carried out with  $J = 10$ ; 12 Terms in Expansions.

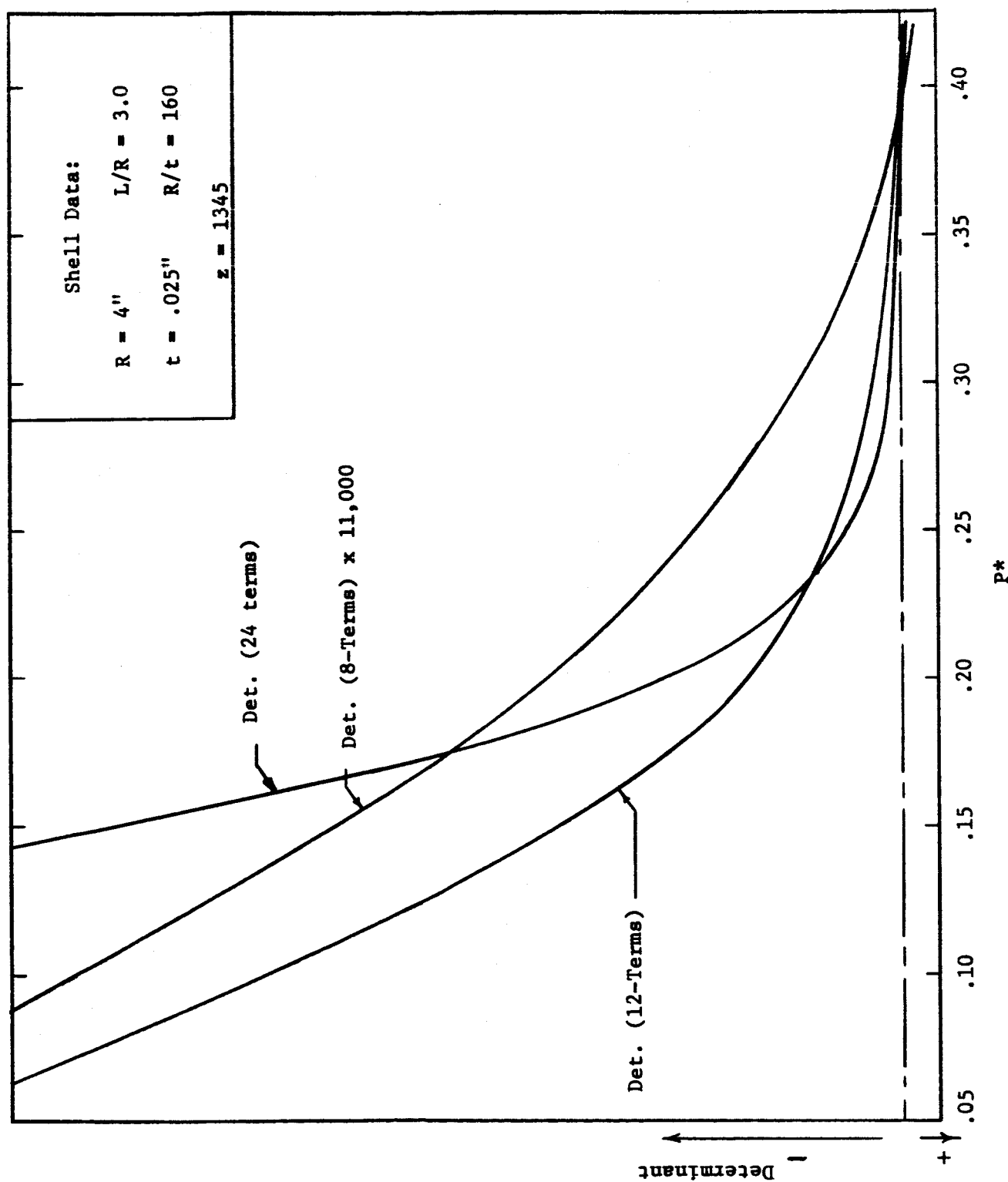


Fig. 20a. Determinant vs.  $P^*$  for 8 and 12 Term Expansions with  $J = 8$  and  $p = 0$ .

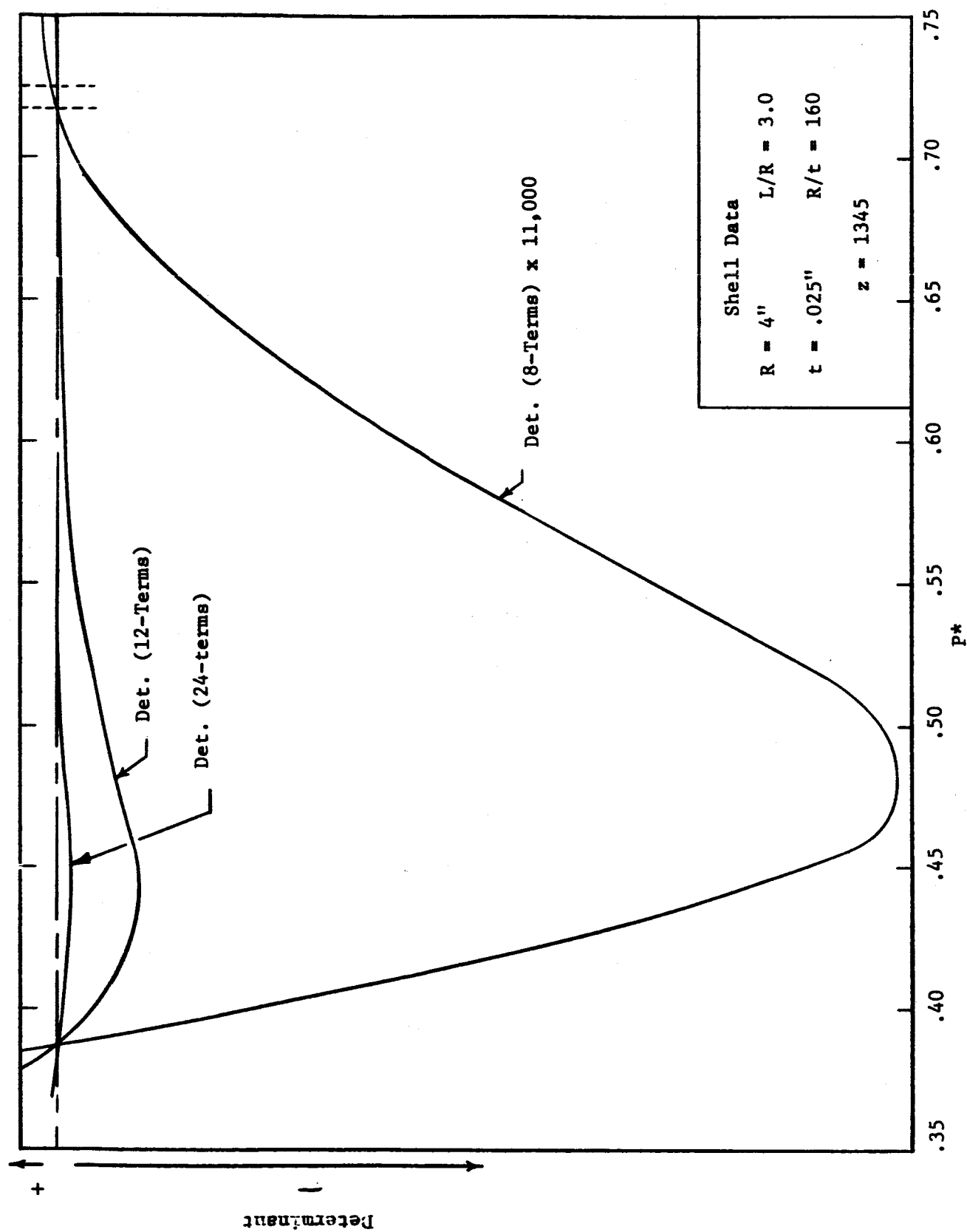


Fig. 20b. Determinant vs.  $P^*$  for 8 and 12 Term Expansions with  $J = 8$  and  $p = 0$ .

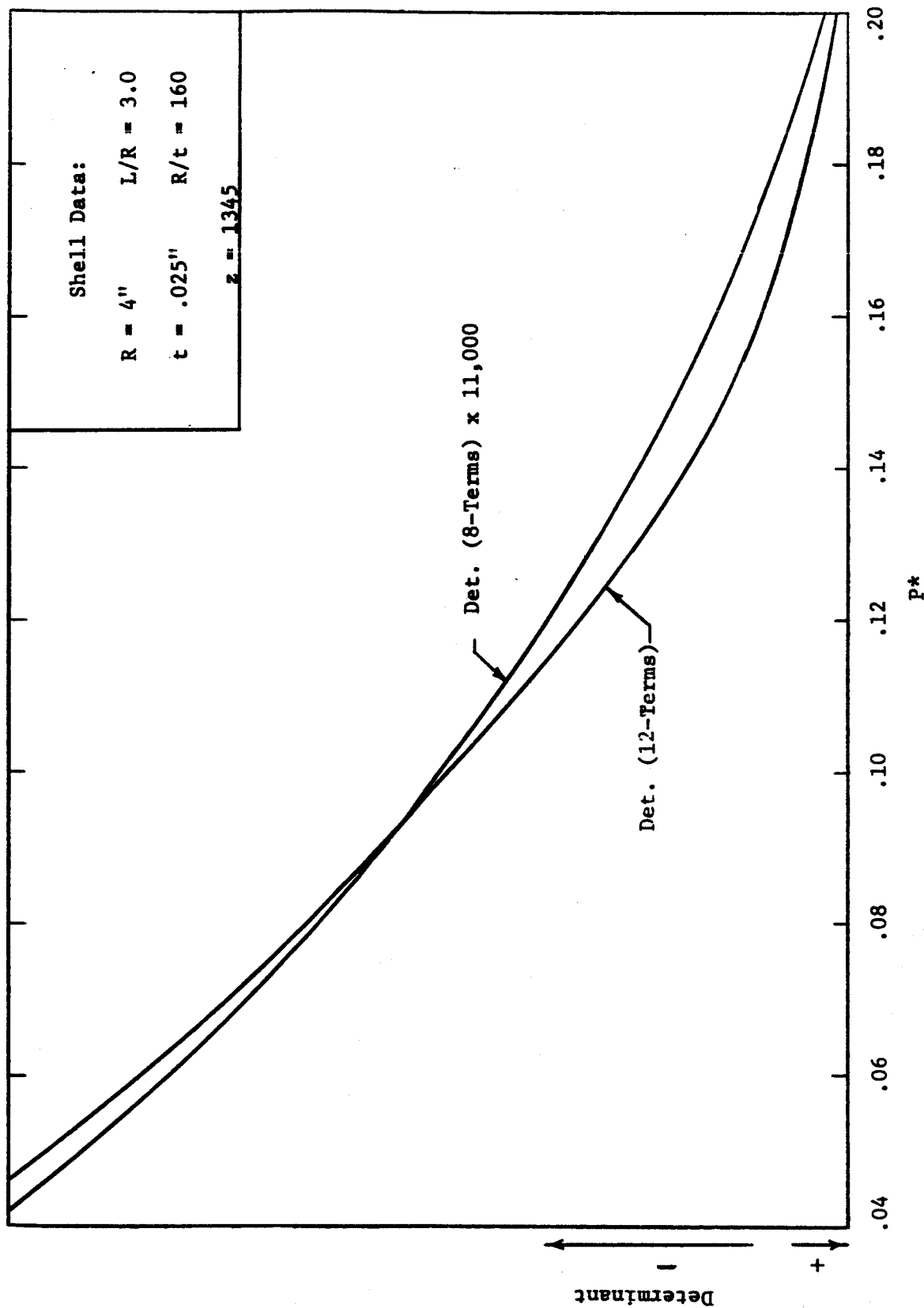


Fig. 21a. Determinant vs.  $P^*$  for 8 and 12 Term Expansions with  $J = 10$  and  $p = 0$ .

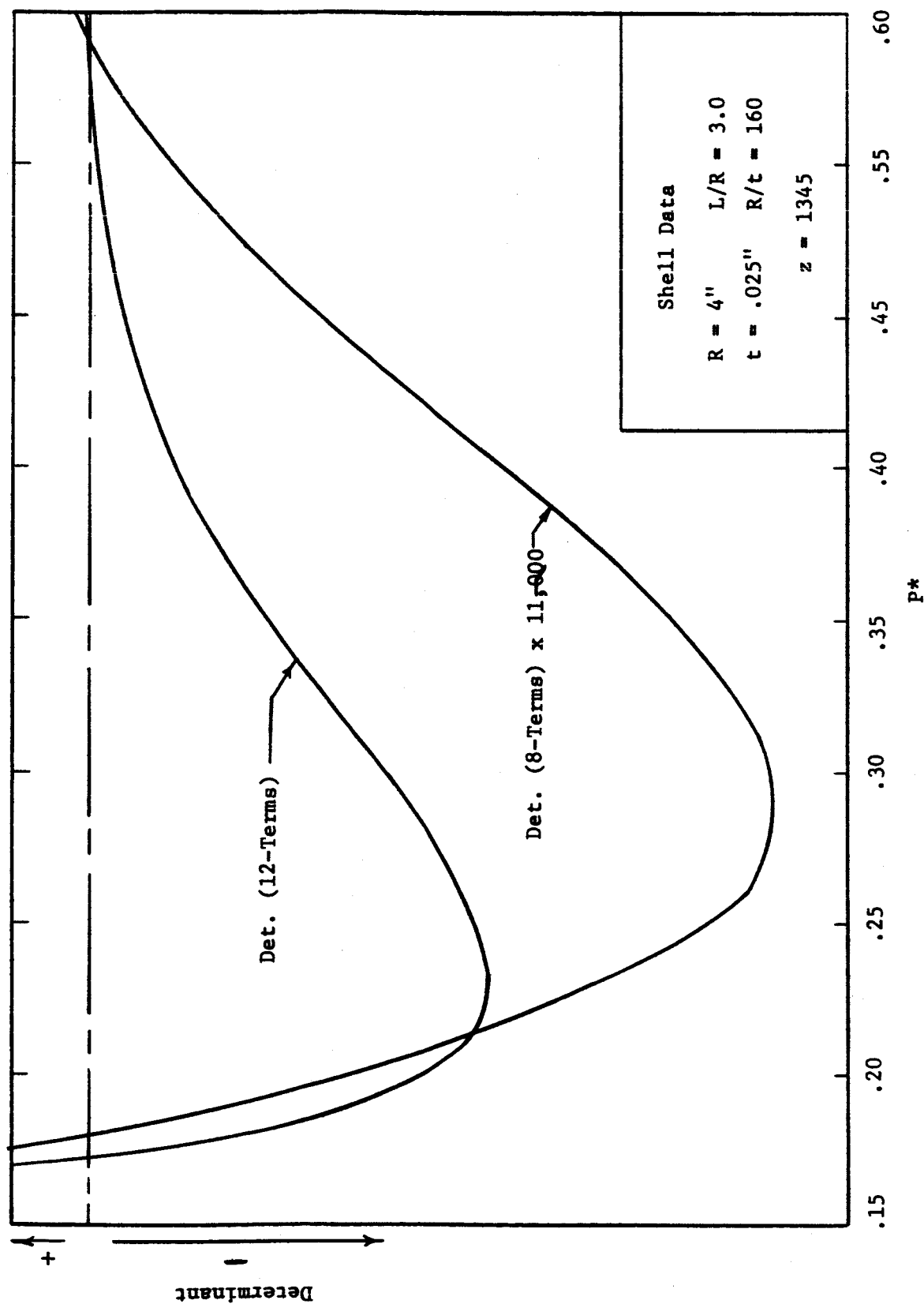


Fig. 21b. Determinant vs.  $P^*$  for 8 and 12 Term Expansions with  $J = 10$  and  $p = 0$ .

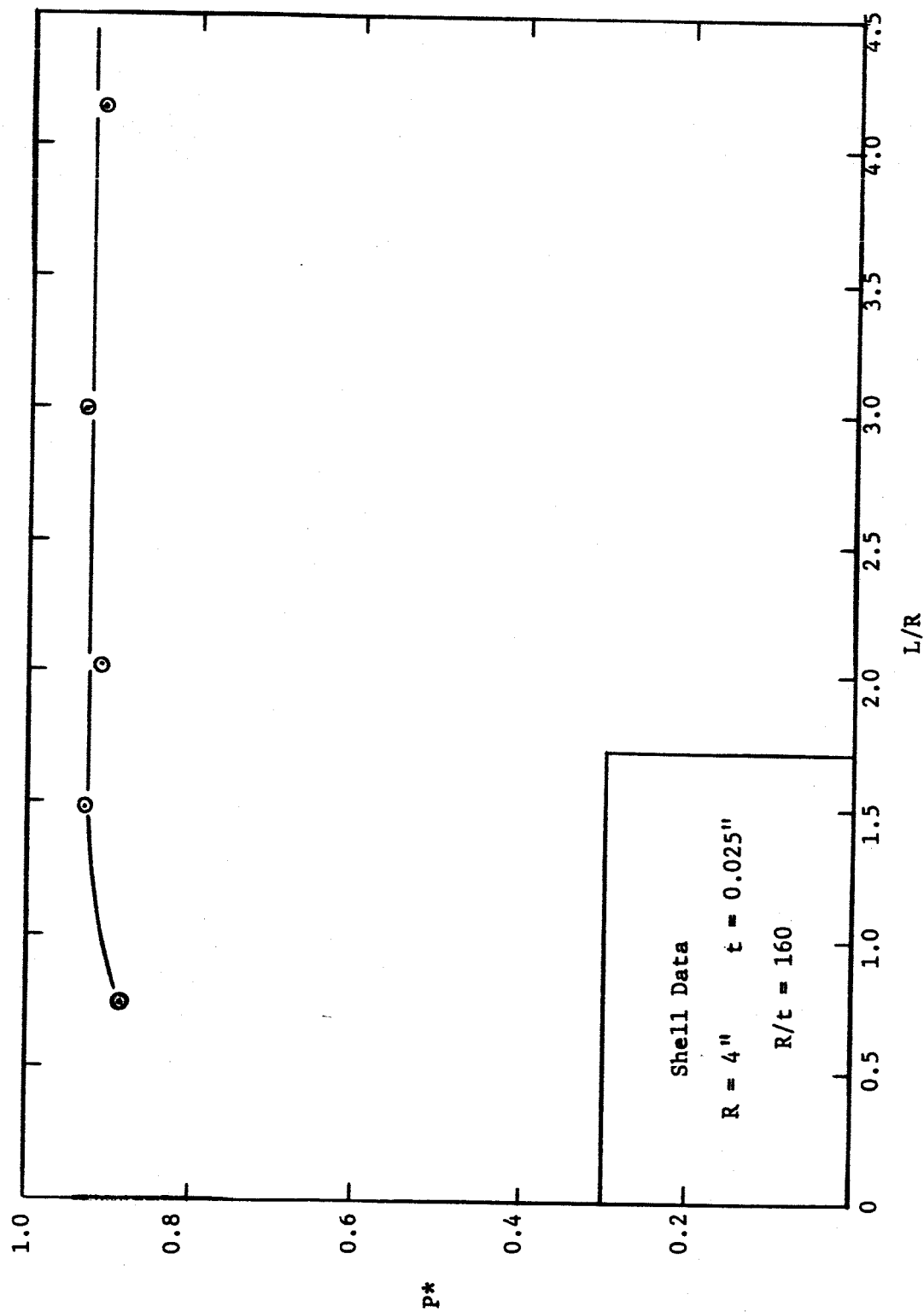


Fig. 22. Experimental Buckling Load vs. Ratio of Length to Radius for an Unpressurized Shell of Fixed Thickness and Length.

## APPENDIX A

Investigation of Number of Terms Required in Expansion

Results of the computations carried out in this paper indicate that, for longer shells subjected to no lateral loading, the analytic results deviate from experiment when the number of peripheral wave permitted is in the neighborhood of 10. Since the size of the matrices used herein was restricted to  $72 \times 72$  one is led to investigate the possible effects of using larger matrices. In the finite difference methods used in Ref. [15] and [23] matrices of not less than  $150 \times 150$  were employed when analysing the behavior of such shells.

As discussed earlier, since prebuckling deformation wavelengths are independent of shell length it is therefore to be expected that more terms in the buckling displacement expansions and hence larger matrices are to be required when analysing larger shells. In Fig. 19 the analytically predicted buckling load vs. ratio of shell length to radius, is plotted for a shell of fixed  $R/t$ , with  $J$  held constant at 10. We observed that the deviation from experiment is relatively small for  $L/R = 0.75$  but increases rapidly as  $L/R$  increases. This observation is consistent with the contention that more terms in the expansions are required for larger shells, especially if a wide range of values of  $J$  are to be investigated.

The determinant vs. loading for  $J = 8$  and  $J = 10$ , with different numbers of terms employed, has been plotted in Figs. 20a, 20b, 21a, and 21b, for a particular shell geometry with  $p = 0$ . In Figs. 20a, and 21a, the determinants have been scaled to give approximately the same magnitude and are plotted from



$P^* = 0.05$  up to the first crossing of the axis. In Figs. 20b, and 21b, these determinants are plotted with their magnitudes in the same ratio as in the corresponding previous figures. The scale has been enlarged for clarity and the value of  $P^*$  varies between the values associated with the first and second crossing of the axis.

We note in these figures that for the 12 term expansion the "dip" below the axis is much less than for the expansion of 8 terms. This would appear to indicate that with sufficient terms taken the "dip" would pull completely above the axis and hence remove the two lowest zeros from the results. The analysis would then give fair agreement with experiment.

## APPENDIX B

Fortran-Pitt Computer Program (Print-out)

•• H MOUNT SCRATCH ON DRIVE 24 WITH RING ON  
•• T THIS PROGRAM LOADS SCRATCH TAPE ON DR 24 FOR NEXT PROG,M  
• COMPILE FORTRAN,EXECUTE FORTRAN,DUMP IF ERROR

66

```
SUBROUTINE CADD(AR,AI,BR,BI,CR,CI)
  CR DEFINED BUT NOT USED IN AN ARITH STMT.
  CI DEFINED BUT NOT USED IN AN ARITH STMT.
SUBROUTINE CSUBT(AR,AI,BR,BI,CR,CI)
  CR DEFINED BUT NOT USED IN AN ARITH STMT.
  CI DEFINED BUT NOT USED IN AN ARITH STMT.
SUBROUTINE CMULT(AR,AI,BR,BI,CR,CI)
  CR DEFINED BUT NOT USED IN AN ARITH STMT.
  CI DEFINED BUT NOT USED IN AN ARITH STMT.
SUBROUTINE CDIV(AR,AI,BR,BI,CR,CI)
  CR DEFINED BUT NOT USED IN AN ARITH STMT.
  CI DEFINED BUT NOT USED IN AN ARITH STMT.
SUBROUTINE SINH(AR,AI,BR,BI)
SUBROUTINE COSH(AR,AI,BR,BI)
SUBROUTINE EZ(AR,AI,BR,BI)
  BR DEFINED BUT NOT USED IN AN ARITH STMT.
  BI DEFINED BUT NOT USED IN AN ARITH STMT.
SUBROUTINE CSCS(A1,A2,A3,A4,EL,C1,S1,VAL)
  VAL DEFINED BUT NOT USED IN AN ARITH STMT.
SUBROUTINE SSCS(A1,A2,A3,A4,EL,C1,S1,VAL)
  VAL DEFINED BUT NOT USED IN AN ARITH STMT.
SUBROUTINE SCC(A1,A2,A3,A4,EL,C1,S1,VAL)
  VAL DEFINED BUT NOT USED IN AN ARITH STMT.
SUBROUTINE CSCC(A1,A2,A3,A4,EL,C1,S1,VAL)
  VAL DEFINED BUT NOT USED IN AN ARITH STMT.
SUBROUTINE CCCC(A1,A2,A3,A4,EL,C1,S1,VAL)
  VAL DEFINED BUT NOT USED IN AN ARITH STMT.
SUBROUTINE SSSS(A1,A2,A3,A4,EL,C1,S1,VAL)
  VAL DEFINED BUT NOT USED IN AN ARITH STMT.
*** MAIN PROGRAM ***
  K DEFINED BUT NOT USED IN AN ARITH STMT.
2000   SUBROUTINE CADD(AR,AI,BR,BI,CR,CI)
2006   CR=AR+BR
2009   CI=AI+BI
2012   RETURN
      0005865028
      END
2033   SUBROUTINE CSUBT(AR,AI,BR,BI,CR,CI)
2039   CR=AR-BR
2042   CI=AI-BI
2045   RETURN
      END
2066   SUBROUTINE CMULT(AR,AI,BR,BI,CR,CI)
2072   A1=AR
2074   A2=AI
2076   B1=BR
2078   B2=BI
2080   CR=A1*B1-A2*B2
2087   CI=A1*B2+A2*B1
2094   RETURN
      END
2120   SUBROUTINE CDIV (AR,AI,BR,BI,CR,CI)
2126   A1=AR
2128   A2=AI
2130   B1=BR
```

```

2132      B2=BI
2134      D=BR*BR+BI*BI
2141      IF (D) 1,2,1
2144      2 PRINT 3
2148      3 FORMAT (12HCDIV BY ZERO)
2151      1 CR=(A1*BI+A2*B2)/D
2161      CI=(A2*BI-A1*B2)/D
      END

2198      SUBROUTINE SINH (AR,AI,BR,BI)
2204      CALL EZ(AR,AI,CR,CI)
2208      CALL EZ(-AR,-AI,DR,DI)
2218      CALL CSUBT(CR,CI,DR,DI,BR,BI)
2226      BR=BR/2.
2230      BI=BI/2.
2234      RETURN
      END

2259      SUBROUTINE COSH(AR,AI,BR,BI)
2265      CALL EZ(AR,AI,CR,CI)
2269      DR=-AR
2271      DI=-AI
2273      CALL EZ(DR,DI,BR,BI)
2277      CALL CADD(CR,CI,BR,BI,BR,BI)
2283      BR=BR/2.
2287      BI=BI/2.
2291      RETURN
      END

2314      SUBROUTINE EZ(AR,AI,BR,BI)
2320      EAR=EXPEF(AR)
2323      BR=EAR*COSF(AI)
2327      BI=EAR*SINF(AI)
2331      RETURN
      END

2351      SUBROUTINE CSCS(A1,A2,A3,A4,EL,C1,S1,VAL)
2357      X=A2+A3-A4
2361      W1=(C1*X*SINF(X*EL))/((A1*A1)+(X*X))
2376      W2=(S1*A1*COSF(X*EL))/((A1*A1)+(X*X))
2391      X=A2+A3+A4
2395      W3=((C1*X*SINF(X*EL))+(S1*A1*COSF(X*EL)))/((A1*A1)+(X*X))
2417      X=A2-A3-A4
2421      W4=((C1*X*SINF(X*EL))+(S1*A1*COSF(X*EL)))/((A1*A1)+(X*X))
2443      X=A2-A3+A4
2447      W5=((C1*X*SINF(X*EL))+(S1*A1*COSF(X*EL)))/((A1*A1)+(X*X))
2469      VAL=(W1+W2-W3+W4-W5)/4.0
2477      RETURN
      END

2508      SUBROUTINE SSCS(A1,A2,A3,A4,EL,C1,S1,VAL)
2514      X=A2+A3-A4
2518      W1=((S1*X*SINF(X*EL))+(C1*A1*COSF(X*EL)))-(A1)/((A1*A1)+(X*X)
1))
2545      X=A2+A3+A4
2549      W2=((S1*X*SINF(X*EL))+(C1*A1*COSF(X*EL)))-(A1)/((A1*A1)+(X*X)
1)
2576      X=A2-A3-A4
2580      W3=((S1*X*SINF(X*EL))+(C1*A1*COSF(X*EL)))-(A1)/((A1*A1)+(X*X)
1)
2607      X=A2-A3+A4
2611      W4=((S1*X*SINF(X*EL))+(C1*A1*COSF(X*EL)))-(A1)/((A1*A1)+(X*X))

```

```

2638 VAL=(W1-W2+W3-W4)/(4.)
2647 RETURN
END
2678 SUBROUTINE SSCC(A1,A2,A3,A4,EL,C1,S1,VAL)
2684 X=A2+A3-A4
2688 W1=((C1*A1*SINF(X*EL))-(S1*X*COSF(X*EL)))/((A1*A1)+(X*X))
2712 X=A2+A3+A4
2716 W2=((C1*A1*SINF(X*EL))-(S1*X*COSF(X*EL)))/((A1*A1)+(X*X))
2740 X=A2-A3-A4
2744 W3=((C1*A1*SINF(X*EL))-(S1*X*COSF(X*EL)))/((A1*A1)+(X*X))
2768 X=A2-A3+A4
2772 W4=((C1*A1*SINF(X*EL))-(S1*X*COSF(X*EL)))/((A1*A1)+(X*X))
2796 VAL=(W1+W2+W3+W4)/(4.)
2805 RETURN
END
2835 SUBROUTINE CSCC(A1,A2,A3,A4,EL,C1,S1,VAL)
2841 X=A2-A3-A4
2845 W1=((S1*A1*SINF(X*EL))-(C1*X*COSF(X*EL))+(X))/((A1*A1)+(X*X))
2870 X=A2-A3+A4
2874 W2=((S1*A1*SINF(X*EL))-(C1*X*COSF(X*EL))+(X))/((A1*A1)+(X*X))
2899 X=A2+A3-A4
2903 W3=((S1*A1*SINF(X*EL))-(C1*X*COSF(X*EL))+(X))/((A1*A1)+(X*X))
2928 X=A2+A3+A4
2932 W4=((S1*A1*SINF(X*EL))-(C1*X*COSF(X*EL))+(X))/((A1*A1)+(X*X))
2957 VAL=(W1+W2+W3+W4)/(4.)
2966 RETURN
END
2996 SUBROUTINE CCCC(A1,A2,A3,A4,EL,C1,S1,VAL)
3002 X=A2-A3+A4
3006 W1=((C1*X*SINF(X*EL))+(S1*A1*COSF(X*EL)))/((A1*A1)+(X*X))
3028 X=A2+A3-A4
3032 W2=((C1*X*SINF(X*EL))+(S1*A1*COSF(X*EL)))/((A1*A1)+(X*X))
3054 X=A2-A3-A4
3058 W3=((C1*X*SINF(X*EL))+(S1*A1*COSF(X*EL)))/((A1*A1)+(X*X))
3080 X=A2+A3+A4
3084 W4=((C1*X*SINF(X*EL))+(S1*A1*COSF(X*EL)))/((A1*A1)+(X*X))
3106 VAL=(W1+W2+W3+W4)/(4.)
3115 RETURN
END
3145 SUBROUTINE SSSS(A1,A2,A3,A4,EL,C1,S1,VAL)
3151 X=A2-A3-A4
3155 W1=((C1*A1*SINF(X*EL))-(S1*X*COSF(X*EL)))/((A1*A1)+(X*X))
3179 X=A2+A3-A4
3183 W2=((C1*A1*SINF(X*EL))-(S1*X*COSF(X*EL)))/((A1*A1)+(X*X))
3207 X=A2-A3+A4
3211 W3=((C1*A1*SINF(X*EL))-(S1*X*COSF(X*EL)))/((A1*A1)+(X*X))
3235 X=A2+A3+A4
3239 W4=((C1*A1*SINF(X*EL))-(S1*X*COSF(X*EL)))/((A1*A1)+(X*X))
3263 VAL=(-W1+W2+W3-W4)/(4.)
3272 RETURN
END

```

CYL SHELL BUC PROB

GALLERKIN METHOD  
 DIMENSION A(24,24)  
 K=24

3303

```

3306      REWIND 15
3308      READ 450,POI,RAD,E,EL,T,R1,P1,PO,DELP,RO,PR,DPR,PRO
3326 450  FORMAT(7(F10.2,1X))
3328      PI=3.141593
3331      D=(E*(T*T*T))/((12.*(1.-(POI*POI)))
3345 341  R=R1
3347      FK1=10000.0
3349      1 P=P1
3351 151  FK=10.0
3353      DO 4 M=1,K
3357      DO 4 N=1,K
3361      IF(M-N)3,2,3
3365      2 FN=N
3368      A(M,N)= (((EL/2.)*((POI-1.0)/2.)*R*R)-((((FN*PI)/EL)**2)*(EL/2.0))
1 )/FK
3399      GO TO 4
3400      3 A(M,N)=0.0
3406      4 CONTINUE
3408      WRITE TAPE 15,((A(I,J),I=1,K),J=1,K)
3427      DO 5 M=1,K
3431      DO 5 N=1,K
3435      FN=N
3438      FM=M
3441      X1=((1.+POI)*R*((2.*FN)-1.))/(4.)
3454      X2=(2.*FN)-1.-(2.*FM)
3464      X3=(2.*FN)-1.+(2.*FM)
3472      5 A(M,N)=((X1*(SINF(X2*PI*.5))/X2)-(X1*(SINF(X3*PI*.5))/X3)
1 )/FK
3499      WRITE TAPE 15,((A(I,J),I=1,K),J=1,K)
3518      DO 6 M=1,K
3522      DO 6 N=1,K
3526      FN=N
3529      FM=M
3532      X1=(1.+POI)*R*FN*.5
3538      X2=(2.*FN)-(2.*FM)+1.0
3548      X3=(2.*FN)+(2.*FM)-1.0
3556      6 A(M,N)=((X1*SINF(X2*PI*.5)/X2)+(X1*SINF(X3*PI*.5)/X3) )/FK
3581      WRITE TAPE 15,((A(I,J),I=1,K),J=1,K)
3600      DO 9 M=1,K
3604      DO 9 N=1,K
3608      FN=N
3611      IF (M-N) 18,7,18
3614 18  A(M,N)=0.0
3620      GO TO 9
3621      7 X1=(EL/2.)*R*R
3627      X2=(EL/2.)*(1.-POI)*.5*(PI/(2.*EL))*(PI/(2.*EL))
1 *(((2.*FN)-1.)**2)
3655      8 A(M,N)=(-X1-X2) /FK
3664      9 CONTINUE
3666      WRITE TAPE 15,((A(I,J),I=1,K),J=1,K)
3685      Z=((EL*EL*4.)/(RAD*T))*(SQRTF(1.-(POI*POI)))
3702      G1=(.25/EL)*(SQRTF(((4.*SQRTF(3.))*Z)+((P*EL*EL*4.)/D)))
3721      G2=(.25/EL)*(SQRTF(((4.*SQRTF(3.))*Z)-((P*EL*EL*4.)/D)))
3742      Q=((RAD*RAD)/(E*T))*(PR+((POI*P)/RAD))
3757      G22=G2*EL
3760      G33=2.*G22
3763      CALL SINH(G22,0,C16,DUMMY)

```

```

3769 CALL COSH(G22,0,C15,DUMMY)
3775 CALL SINH(G33,0,C11,DUMMY)
3781 Q1=(G1*C11)+(2.*G2*SINF(G1*EL)*COSF(G1*EL))
3796 A11=(((-2.)*Q)*((G1*SINF(G1*EL)*C15)-(G2*COSF(G1*EL)*C16)))/(Q1)
3821 A22=(((-2.)*Q)*((G2*SINF(G1*EL)*C15)+(G1*COSF(G1*EL)*C16)))/(Q1)
3844 GA1=(A11*G2)-(A22*G1)
3853 GA2=(A22*G2)+(A11*G1)
3860 GA3=(GA1*G2)-(GA2*G1)
3869 GA4=(GA2*G2)+(GA1*G1)
3876 S1=C16
3878 C1=C15
3880 DO 81 M=1,K
3884 DO 81 N=1,K
3888 FN=N
3891 FM=M
3894 IF(M-N+1) 75,76,75
3898 75 W1=0.0
3900 GO TO 77
3901 76 W1=(.5*(1.-FN)*PI*POI)/RAD
3909 77 IF(M-N)78,79,78
3912 78 W2=0.0
3914 GO TO 80
3915 79 W2=(.5*POI*FN*PI*(-1.))/RAD
3923 80 X1=((FN-1.)*PI)/EL
3929 X2=(FM*PI)/EL
3934 CALL CSCS(G2,G1,X1,X2,EL,C1,S1,X3)
3944 W3=-(((GA1*(FN-1.)*(FN-1.)*PI*PI)/(EL*EL))*X3)-((1.-POI)*
1 .5*R*R*GA1*X3)
3971 X1=(FN-1.)*PI/EL
3979 X2=FM*PI/EL
3984 CALL SSCC(G2,X2,G1,X1,EL,C1,S1,X3)
3994 X4=GA2*(((FN-1.)*PI)/EL)**2)
4003 X5=(1.-POI)*.5*R*R*GA2
4010 W4=(-1.)*(X4+X5)*X3
4017 X1=FN*PI/EL
4022 X2=FM*PI/EL
4027 CALL CSCS(G2,G1,X1,X2,EL,C1,S1,X3)
4037 X4=GA1*(((FN*PI/EL)**2)
4045 X5=(1.-POI)*.5*R*R*GA1
4052 W5=(-X4-X5)*X3
4056 CALL SSCC(G2,X2,G1,X1,EL,C1,S1,X3)
4066 X4=GA2*(((FN*PI/EL)**2)
4074 X5=(1.-POI)*.5*R*R*GA2
4081 W6=(-X4-X5)*X3
4085 X1=(FN-1.)*PI/EL
4093 X2=FM*PI/EL
4098 CALL SSSS(G2,G1,X1,X2,EL,C1,S1,X3)
4108 W7=((1.-FN)*PI*GA3/EL)*X3
4118 CALL CSCS(G2,X1,G1,X2,EL,C1,S1,X3)
4128 W8=((1.-FN)*PI*GA4/EL)*X3
4138 X1=FN*PI/EL
4143 CALL SSSS(G2,G1,X1,X2,EL,C1,S1,X3)
4153 W9=(((-FN)*PI*GA3/EL)*X3
4162 CALL CSCS(G2,X1,G1,X2,EL,C1,S1,X3)
4172 W10=(((-FN)*PI*GA4/EL)*X3
4181 81 A(M,N)=(W1+W2+W3+W4+W5+W6+W7+W8+W9+W10 )/FK
4200 WRITE TAPE 15,((A(I,J),I=1,K),J=1,K)

```

```

4219 DO 85 M=1,K
4223 DO 85 N=1,K
4227 FN=N
4230 FM=M
4233 X1=((2.*FN)-(2.*FM)-(1.))*PI
4246 X2=((2.*FN)+(2.*FM)-(3.))*PI
4259 W1=(R/RAD)*((SINF(X1/2.)/(X1/EL))+(SINF(X2/2.)/(X2/EL)))
4287 X1=(FN-FM+.5)*PI
4292 X2=(FN+FM-.5)*PI
4297 W2=(EL*R/RAD)*((SINF(X1)/(2.*X1))+(SINF(X2)/(2.*X2)))
4320 X1=(FN-1.)*PI/EL
4328 X2=((2.*FM)-1.)*PI/(2.*EL)
4338 CALL SSCC(G2,G1,X1,X2,EL,C1,S1,X3)
4348 W3=(1.-POI)*.5*R*GA3*X3
4355 X1=FN*PI/EL
4360 CALL SSCC(G2,G1,X1,X2,EL,C1,S1,X3)
4370 W4=(1.-POI)*.5*R*GA3*X3
4377 X1=(FN-1.)*PI/EL
4385 CALL CCCC(G2,G1,X1,X2,EL,C1,S1,X3)
4395 W5=(1.-POI)*.5*R*GA4*X3
4402 X1=FN*PI/EL
4407 CALL CCCC(G2,G1,X1,X2,EL,C1,S1,X3)
4417 W6=(1.-POI)*.5*R*GA4*X3
4424 X1=(FN-1.)*PI/EL
4432 CALL CSCS(G2,G1,X2,X1,EL,C1,S1,X3)
4442 W7=((1.+POI)*.5*R*(1.-FN)*PI*GA1/EL)*X3
4458 CALL SSCC(G2,X1,G1,X2,EL,C1,S1,X3)
4468 W8=((1.+POI)*.5*R*(1.-FN)*PI*GA2/EL)*X3
4484 X1=FN*PI/EL
4489 CALL CSCS(G2,G1,X2,X1,EL,C1,S1,X3)
4499 W9=(((-1.)*(1.+POI)*.5*R*FN*PI*GA1)/EL)*X3
4513 CALL SSCC(G2,X1,G1,X2,EL,C1,S1,X3)
4523 W10=(((-1.)*(1.+POI)*.5*R*FN*PI*GA2)/EL)*X3
85 A(M,N)=(W1+W2+W3+W4+W5+W6+W7+W8+W9+W10)/FK
4556 WRITE TAPE 15,((A(I,J)),I=1,K),J=1,K)
4575 FK=FK1
4577 DO 90 M=1,K
4581 DO 90 N=1,K
4585 FN=N
4588 FM=M
4591 IF(M-N)87,86,87
4594 87 IF(M-N-1)88,86,88
4599 86 W1=(POI*E*T*FN*PI)/(2.*RAD*(1.-(POI*POI)))
4615 GO TO 89
4616 88 W1=0.0
4618 89 X1=FN*PI/EL
4623 X2=(FM-1.)*PI/EL
4631 CALL SSCC(G2,G1,X1,X2,EL,C1,S1,X3)
4641 X2=FM*PI/EL
4646 CALL SSCC(G2,G1,X1,X2,EL,C1,S1,X4)
4656 W2=(((-1.)*(E*T*FN*PI*GA3))/((1.-(POI*POI))*EL)
4674 W2=W2*(X3+X4)
4678 CALL CCCC(G2,G1,X1,X2,EL,C1,S1,X3)
4688 X2=((FM-1.)*PI)/EL
4694 CALL CCCC(G2,G1,X1,X2,EL,C1,S1,X4)
4704 W3=(((-1.)*E*T*GA4*FN*PI)/((1.-(POI*POI))*EL)
4720 W3=W3*(X3+X4)

```



```

4724 90 A(M ,N)=(W1+W2+W3)/FK
4736 WRITE TAPE 15,((A(I,J),I=1,K),J=1,K)
4755 DO 91 M=1,K
4759 DO 91 N=1,K
4763 FN=N
4766 FM=M
4769 X1=(2.*FM)-(2.*FN)-1.
4779 W1=(SINF((.5)*PI*X1))/(X1*PI/EL)
4791 X2=(2.*FM)+(2.*FN)-3.
4799 W2=(SINF(PI*.5*X2))/(X2*PI/EL)
4811 X1=FM-FN+.5
4815 W3=(SINF(PI*X1))/(PI*2.*X1/EL)
4829 X1=FM+FN-.5
4833 W4=(SINF(PI*X1))/(PI*2.*X1/EL)
4847 W11=((-1.)*E*T*R)/(RAD*(1.-(POI*POI)))
4861 W1=W11*(W1+W2+W3+W4)
4867 X1=((2.*FN)-1.)*PI/(2.*EL)
4877 X2=(FM-1.)*PI/EL
4885 CALL SSCC(G2,G1,X1,X2,EL,C1,S1,X3)
4895 X2=FM*PI/EL
4900 CALL SSCC(G2,G1,X1,X2,EL,C1,S1,X4)
4910 W2=(E*T*POI*R*GA3*(X3+X4))/(1.-(POI*POI))
4926 CALL CCCC(G2,G1,X1,X2,EL,C1,S1,X3)
4936 X2=(FM-1.)*PI/EL
4944 CALL CCCC(G2,G1,X1,X2,EL,C1,S1,X4)
4954 W3=(E*T*POI*R*GA4*(X3+X4))/(1.-(POI*POI))
4970 91 A(M , N)=(W1+W2+W3)/FK
4982 WRITE TAPE 15,((A(I,J),I=1,K),J=1,K)
5001 G22=2.*G2
5004 G11=2.*G1
5007 GL2=G22*EL
5010 CALL COSH (GL2,0.0,C11,DUMMY)
5016 CALL SINH (GL2,0.0,S11,DUMMY)
5022 C111=C1
5024 S111=S1
5026 DO 131 M=1,K
5030 DO 131 N=1,K
5034 FN=N
5037 FM=M
5040 IF (M-N) 93,92,93
5043 93 IF(M-N+1)94,92,94
5048 92 W1=(((FN-1.)*PI)/EL)**4)*D*EL*.5
5059 GO TO 95
5060 94 W1=0.0
5062 95 IF(M-N)97,96,97
5065 97 IF(M-N-1)98,96,98
5070 96 W2=((FN*PI/EL)**4)*D*EL*.5
5080 GO TO 99
5081 98 W2=0.0
5083 99 IF(M-N)101,100,101
5086 101 IF(M-N+1)102,100,102
5091 100 W3=2.*R*R*D*EL*.5*(((FN-1.)*PI/EL)**2)
5107 GO TO 103
5108 102 W3=0.0
5110 103 IF(M-N)105,104,105
5113 105 IF(M-N-1)106,104,106
5118 104 W4=2.*R*R*D*EL*.5*FN*FN*PI*PI/(EL*EL)

```

```

5132      GO TO 107
5133 106 W4=0.0
5135 107 X1=(E*T)/((1.-(POI-POI))*RAD*RAD)
5148      X2=D*(R**4)
5153      X1=X1+X2
5156      CALL SSCC(G22,X1,0.0,X2,EL,C1,S1,X3)
5166 108 IF(M-N)110,109,110
5170 109 W5=X1*EL
5173      GO TO 111
5174 110 W5=0.0
5176 111 IF(M-N+1)113,112,113
5180 113 IF(M-N-1)114,112,114
5185 112 W5A=X1*EL*.5
5189      GO TO 115
5190 114 W5A=0.0
5192 115 Y1=(      POI*E*T*GA1*PI)/(RAD*(1.-(POI*POI)))
5207      X1=(FN-1.)*PI/EL
5215      X2=(FM-1.)*PI/EL
5223      CALL CSCS(G2,G1,X2,X1,EL,C1,S1,X3)
5233      X2=FM*PI/EL
5238      CALL CSCS(G2,G1,X2,X1,EL,C1,S1,X4)
5248      X1=FN*PI/EL
5253      X2=(FM-1.)*PI/EL
5261      CALL CSCS(G2,G1,X2,X1,EL,C1,S1,X5)
5271      X2=FM*PI/EL
5276      CALL CSCS(G2,G1,X2,X1,EL,C1,S1,X6)
5286      W6=((1.-FN)/EL)*Y1*(X3+X4))-(((FN      /EL)*Y1)*(X5+X6))
5308      Y=E*(-1.)*POI*T*GA2*PI/(RAD*(1.-(POI*POI))*EL)
5327      X1=(FN-1.)*PI/EL
5335      X2=(FM-1.)*PI/EL
5343      CALL SSCC(G2,X1,G1,X2,EL,C1,S1,X3)
5353      X2=FM*PI/EL
5358      CALL SSCC(G2,X1,G1,X2,EL,C1,S1,X4)
5368      X1=FN*PI/EL
5373      X2=(FM-1.)*PI/EL
5381      CALL SSCC(G2,X1,G1,X2,EL,C1,S1,X5)
5391      X2=FM*PI/EL
5396      CALL SSCC(G2,X1,G1,X2,EL,C1,S1,X6)
5406      W7=((FN-1.)*Y*(X3+X4))+(FN*Y*(X5+X6))
5420      IF(M-N)117,116,117
5423 117 IF(M-N+1)118,116,118
5428 116 W8=(-P)*(((FN-1.)*PI)/EL)**2)*EL*.5
5441      GO TO 119
5442 118 W8=0.0
5444 119 IF(M-N)121,120,121
5447 121 IF(M-N-1)122,120,122
5452 120 W9=(-P)*((FN*PI/EL)**2)*EL*.5
5464      GO TO 123
5465 122 W9=0.0
5467 123 IF(M-N)125,124,125
5471 124 W10=(E*T*R*R*Q*EL/RAD)-(POI*P*R*R*EL)
5491      GO TO 126
5492 125 W10=0.0
5494 126 IF(M-1-N)128,127,128
5498 128 IF(M+1-N)129,127,129
5503 127 W10A=((E*T*R*R*Q/RAD)-(POI*P*R*R))*EL*.5
5523      GO TO 130

```

```

5524 129 W10A=0.0
5526 130 C1=C11
5528     S1=S11
5530     X1=(FN-1.)*PI/EL
5538     X2=(FM-1.)*PI/EL
5546     X2=FM*PI/EL
5551     CALL SSCC(G22,X1,0.0,X2,EL,C1,S1,X4)
5561     CALL SSCC(G22,X1,G11,X2,EL,C1,S1,X6)
5571     X2=(FM-1.)*PI/EL
5579     CALL SSCC(G22,X1,G11,X2,EL,C1,S1,X5)
5589     X10=(E*T*GA1*GA3*PI)/(4.*EL*(1.-(POI*POI)))
5605     W15A=(FN-1.)*X10*(X3+X4-X5-X6)
5615     X1=FN*PI/EL
5620     CALL SSCC(G22,X1,G11,X2,EL,C1,S1,X3)
5630     CALL SSCC(G22,X1,0.0,X2,EL,C1,S1,X5)
5640     X2=FM*PI/EL
5645     CALL SSCC(G22,X1,G11,X2,EL,C1,S1,X4)
5655     CALL SSCC(G22,X1,0.0,X2,EL,C1,S1,X6)
5665     W15B=FN*X10*(X3+X4-X5-X6) *(-1.)
5675     W15=W15A+W15B
5678     X10=E*T*GA2*GA3*PI/(4.*EL*(1.-(POI*POI)))
5692     X1=(FN-1.)*PI/EL
5700     X2=(FM-1.)*PI/EL
5708     CALL CSCS(G22,G11,X2,X1,EL,C1,S1,X3)
5718     CALL CSCS(0.0,G11,X2,X1,EL,1.0,0.0,X5)
5728     X2=(PI/EL)*FM
5733     CALL CSCS(G22,G11,X2,X1,EL,C1,S1,X4)
5743     CALL CSCS(0.0,G11,X2,X1,EL,1.0,0.0,X6)
5753     W16A=(FN-1.)*X10*(X3+X4-X5-X6)
5763     X1=FN*PI/EL
5768     CALL CSCS(G22,G11,X2,X1,EL,C1,S1,X4)
5778     CALL CSCS(0.0,G11,X2,X1,EL,1.0,0.0,X6)
5788     X2=(FM-1.)*PI/EL
5796     CALL CSCS(G22,G11,X2,X1,EL,C1,S1,X3)
5806     CALL CSCS(0.0,G11,X2,X1,EL,1.0,0.0,X5)
5816     W16B=FN*X10*(X3+X4-X5-X6)
5823     W16=W16A+W16B
5826     X10=E*T*GA1*GA4*PI/(4.*EL*(1.-(POI*POI)))
5840     X1=(FN-1.)*PI/EL
5848     CALL CSCS(G22,G11,X2,X1,EL,C1,S1,X3)
5858     CALL CSCS(0.0,G11,X2,X1,EL,1.0,0.0,X5)
5868     X2=FM*PI/EL
5873     CALL CSCS(G22,G11,X2,X1,EL,C1,S1,X4)
5883     CALL CSCS(0.0,G11,X2,X1,EL,1.0,0.0,X6)
5893     W17A=X10*(FN-1.)*(X3+X4+X5+X6)
5903     X1=FN*PI/EL
5908     CALL CSCS(G22,G11,X2,X1,EL,C1,S1,X4)
5918     CALL CSCS(0.0,G11,X2,X1,EL,1.0,0.0,X6)
5928     X2=(FM-1.)*PI/EL
5936     CALL CSCS(G22,G11,X2,X1,EL,C1,S1,X3)
5946     CALL CSCS(0.0,G11,X2,X1,EL,1.0,0.0,X5)
5956     W17B=FN*X10*(X3+X4+X5+X6)
5963     W17=W17A+W17B
5966     X10=E*T*GA2*GA4*PI/(4.*(1.-(POI*POI)))
5979     X1=(FN-1.)*PI/EL
5987     X2=(FM-1.)*PI/EL
5995     CALL SSCC(G22,X1,0.0,X2,EL,C1,S1,X3)

```

```

6005 CALL SSCC(G22,X1,G11,X2,EL,C1,S1,X5)
6015 X2=FM*PI/EL
6020 CALL SSCC(G22,X1,0.0,X2,EL,C1,S1,X4)
6030 CALL SSCC(G22,X1,G11,X2,EL,C1,S1,X6)
6040 W18A=(FN-1.)*X10*(X3+X4+X5+X6) /EL
6052 X1=FN*PI/EL
6057 X2=(FM-1.)*PI/EL
6065 CALL SSCC(G22,X1,0.0,X2,EL,C1,S1,X3)
6075 CALL SSCC(G22,X1,G11,X2,EL,C1,S1,X5)
6085 X2=FM*PI/EL
6090 CALL SSCC(G22,X1,0.0,X2,EL,C1,S1,X4)
6100 CALL SSCC(G22,X1,G11,X2,EL,C1,S1,X6)
6110 W18B=FN*X10*(X3+X4+X5+X6) /EL
6119 W18=W18A+W18B
6122 C1=C111
6124 S1=S111
6126 X10=E*T*R=R*A11/RAD
6136 X1=(FN-1.)*PI/EL
6144 X2=(FM-1.)*PI/EL
6152 CALL SSCC(G2,G1,X1,X2,EL,C1,S1,X3)
6162 X2=FM*PI/EL
6167 CALL SSCC(G2,G1,X1,X2,EL,C1,S1,X4)
6177 X1=FN*PI/EL
6182 CALL SSCC(G2,G1,X1,X2,EL,C1,S1,X6)
6192 X2=(FM-1.)*PI/EL
6200 CALL SSCC(G2,G1,X1,X2,EL,C1,S1,X5)
6210 W11=X10*(X3+X4+X5+X6)
6216 X10=E*T*R=R*A22/RAD
6226 CALL CCCC(G2,G1,X1,X2,EL,C1,S1,X5)
6236 X2=FM*PI/EL
6241 CALL CCCC(G2,G1,X1,X2,EL,C1,S1,X6)
6251 X1=(FN-1.)*PI/EL
6259 CALL CCCC(G2,G1,X1,X2,EL,C1,S1,X4)
6269 X2=(FM-1.)*PI/EL
6277 CALL CCCC(G2,G1,X1,X2,EL,C1,S1,X3)
6287 W12=X10*(X3+X4+X5+X6)
6293 X10=((1.-1.)*E*T*GA3)/(RAD*(1.-(POI*POI)))
6307 CALL SSCC(G2,G1,X1,X2,EL,C1,S1,X3)
6317 X2=FM*PI/EL
6322 CALL SSCC(G2,G1,X1,X2,EL,C1,S1,X4)
6332 X1=FN*PI/EL
6337 X2=(FM-1.)*PI/EL
6345 CALL SSCC(G2,G1,X1,X2,EL,C1,S1,X6)
6355 CALL SSCC(G2,G1,X1,X2,EL,C1,S1,X5)
6365 W13=X10*(X3+X4+X5+X6)*POI
6372 X10=((1.-1.)*E*T*GA4)/(RAD*(1.-(POI*POI)))
6386 CALL CCCC(G2,G1,X1,X2,EL,C1,S1,X5)
6396 X2=FM*PI/EL
6401 CALL CCCC(G2,G1,X1,X2,EL,C1,S1,X6)
6411 X1=(FN-1.)*PI/EL
6419 CALL CCCC(G2,G1,X1,X2,EL,C1,S1,X4)
6429 X2=(FM-1.)*PI/EL
6437 CALL CCCC(G2,G1,X1,X2,EL,C1,S1,X3)
6447 W14=X10*(X3+X4+X5+X6)*POI
6454 131 A(M , N)= (W1+W2+W3+W4+W5+W5A+W6+W7+W8+W9+W10+W10A+
1 W11+W12+W13+W14+W15+W16+W17+W18)/FK
6483 WRITE TAPE 15,((A(I,J),I=1,K),J=1,K)

```

```

6502      P=P+DELP
6505      IF (P-P0) 151,337,151
6509      337 IF (R-R0) 338,339,338
6512      338 R=R+2.0
6515      GO TO 1
6516      339 IF (PR-PR0) 340,336,340
6519      340 PR=PR+DPR
6522      GO TO 341
6523      336 PRINT 470,POI,RAD,E,EL,T,R,PR
6534      END FILE 15
6536      REWIND 15
6539      470 FORMAT (10(F10.2,1X))
6541      STOP
        END

```

# SUBROUTINE CADD

VARIABLES  
NAME LOC.

AI	0000	AR	0000	BI	0000	BR	0000	CI	0000	CR
----	------	----	------	----	------	----	------	----	------	----

STATEMENT NUMBERS  
STMNT LOC.

... NONE ...

# SUBROUTINE CSUBT

VARIABLES  
NAME LOC.

AI	0000	AR	0000	BI	0000	BR	0000	CI	0000	CR
----	------	----	------	----	------	----	------	----	------	----

STATEMENT NUMBERS  
STMNT LOC.

... NONE ...

# SUBROUTINE CMULT

VARIABLES  
NAME LOC.

AI	0000	A1	2115	BI	0000	B1	2117	CI	0000
AR	0000	A2	2116	BR	0000	B2	2118	CR	0000

STATEMENT NUMBERS  
STMNT LOC.

... NONE ...

# SUBROUTINE CDIV

VARIABLES  
NAME LOC.

AI	0000	A1	2192	BI	0000	B1	2194	CI	0000	D
AR	0000	A2	2193	BR	0000	B2	2195	CR	0000	

STATEMENT NUMBERS

\*\* T THIS PROG,M READS SCRATCH TAPE ON DR 24 LOADED BY PROG,M 1  
• COMPILE FORTRAN,EXECUTE FORTRAN, DUMP IF ERROR

77

SUBROUTINEDET(A,N,JX,ANS)

LC DEFINED BUT NOT USED IN AN ARITH STMNT.

I1 DEFINED BUT NOT USED IN AN ARITH STMNT.

I2 DEFINED BUT NOT USED IN AN ARITH STMNT.

MO DEFINED BUT NOT USED IN AN ARITH STMNT.

\*\*\* MAIN PROGRAM \*\*\*

KK DEFINED BUT NOT USED IN AN ARITH STMNT.

JJJ DEFINED BUT NOT USED IN AN ARITH STMNT.

2000 SUBROUTINE DET(A,N,JX,ANS)

DIMENSION A(1300)

2006 LC=N

2008 LR=N

2010 23 DO 31 L=1,LR

2014 NO=L+JX\*(L-1)

2021 3 IF(L-LR)2,4,4

TEST FOR POSSIBLE ROW INTERCHANGE

2026 2 BIGA=A(NO)

2030 NPN=0

2032 I1=L+1

2035 DO 25 JO=I1,LR

2040 NP=JO+JX\*(L-1)

2047 IF(ABSF(BIGA)-ABSF(A(NP)))24,25,25

2056 24 BIGA=A(NP)

2058 NPN=NP

2060 25 CONTINUE

TEST FOR POSSIBLE COLUMN INTERCHANGE

2061 NPM=0

2063 I2=L+1

2066 DO 52 M=I2,LC

2071 NZ = L+JX\*(M-1)

2078 IF(ABSF(BIGA)-ABSF(A(NZ))) 51,52,52

2087 51 NPM=NZ

2089 BIGA=A(NZ)

2091 52 CONTINUE

2092 IF(NPM) 55,54,55

2095 54 IF(NPN) 27,4,27

INTERCHANGE COLUMNS

2098 55 DO 56 K=L,LC

2103 NQ = K+(L-1)\*JX

2109 NU = NPM+(K-L)

2115 C = -A(NQ)

2119 A(NQ) = A(NU)

2123 56 A(NU) = C

2126 GO TO 4

INTERCHANGE ROWS

2127 27 DO 26 K=L,LC

2132 NQ = L+JX\*(K-1)

2139 C=-A(NQ)

2143 NU = NPN-(L-1)\*JX+JX\*(K-1)

2157 A(NQ)=A(NU)

2161 26 A(NU)=C

2164 4 DIVA=1.0/A(NO)

## \*TEST FOR COMPUTATIONAL SINGULARITY

```

2170      IF DIVIDE CHECK 6,11
2172      6 SENSE LIGHT 4
2173      PRINT 16,L,A(NO)
2181      16 FORMAT(12HERROR IN ROW13,21H OF SIMEQ-DIVIDING BY E16.8)
2191      ANS=.99999999E49
2193      RETURN
2194      11 IF(L-LR)12,42,42

```

## MATRIX TRANSFORMATION

```

2199      12 MO=L+1
2202      DO 28 J=MO,LC
2207      NR=L+JX*(J-1)
2214      28 A(NR)=A(NR)*DIVA
2220      29 I1=L+1
2223      DO 31 I=I1,LR
2228      NS=I+JX*(L-1)
2235      FMLTA=A(NS)
2239      DO 31 J=L,LC
2244      NT=I+JX*(J-1)
2251      NY=L+JX*(J-1)
2258      31 A(NT)=A(NT)-A(NY)*FMLTA

```

## COMPUTE THE DETERMINATE = PI OF A(I,I)

```

2269      42 ANS=1.0
2271      DO 44 I=1,N
2275      NV=I+JX*(I-1)
2282      44 ANS=ANS*A(NV)
2288      RETURN
      END

```

## CYL SHELL PROB

## GALERKIN METHOD

```

      DIMENSION A(72,72)
2340      K=24
2343      KK=2*K
2346      KKK=3*K
2349      X=0.0
2351      Y=4.0
2353      9 READ TAPE 15 ,((A(I,J),I=1,K),J=1,K)
2372      JJ=K+1
2375      1 READ TAPE 15, ((A(I,J),I=1,K),J=JJ,KK)
2395      2 READ TAPE 15, ((A(I,J),I=JJ,KK),J=1,K)
2415      3 READ TAPE 15, ((A(I,J),I=JJ,KK),J=JJ,KK)
2436      JJJ=JJ+K
2439      4 READ TAPE 15, ((A(I,J),I=1,K),J=JJJ,KKK)
2459      5 READ TAPE 15, ((A(I,J),I=JJ,KK),J=JJJ,KKK)
2480      6 READ TAPE 15, ((A(I,J),I=JJJ,KKK),J=1,K)
2500      7 READ TAPE 15, ((A(I,J),I=JJJ,KKK),J=JJ,KK)
2521      8 READ TAPE 15, ((A(I,J),I=JJJ,KKK),J=JJJ,KKK)
2542      CALL DET (A,KKK,KKK,ANS)
2548      PRINT 333, ANS
2554      333 FORMAT (2E20.8)
2555      X=X+1.
2559      IF (X-Y) 9,10,9

```

2563 10 REWIND 15  
2585 STOP  
END

79

SUBROUTINE DET

VARIABLES  
NAME LOC.

A	0000	DIVA	2327	I2	2320	K	2323	M	2321	NP
ANS	0000	FMLTA	2333	J	2329	L	2312	MO	2328	NPM
BIGA	2314	I	2331	JO	2317	LC	2310	N	0000	NPN
C	2326	I1	2316	JX	0000	LR	2311	NO	2313	NQ

STATEMENT NUMBERS

STMNT LOC.									
2 2026	6 2172	16 2181	25 2060	28 2214	4.				
3 2021	11 2194	23 2010	26 2161	29 2220	4.				
4 2164	12 2199	24 2056	27 2127	31 2258	5				

MAIN PROGRAM

VARIABLES  
NAME LOC.

A	2568	I	7758	JJ	7759	K	7752	KKK	7754	Y
ANS	7761	J	7757	JJJ	7760	KK	7753	X	7755	

STATEMENT NUMBERS

STMNT LOC.									
1 2375	3 2415	5 2459	7 2500	9 2353	33.				
2 2395	4 2439	6 2480	8 2521	10 2563					

HIGHEST ADDRESS ASSIGNED 7772



## BIBLIOGRAPHY

1. Timoshenko, S. and Gere, J., "Theory of Elastic Stability," McGraw Hill Book Co., 2nd ed., pp. 462-466.
2. Donnell, L. H., "A New Theory for the Buckling of Thin Cylinders under Axial Compression and Bending," ASME Transactions, vol. 56, pp. 795-806.
3. Lundquist, E., "Strength Tests of Thin-Walled Duralumin Cylinders in Compression," NACA Rep. 473, 1933.
4. Donnell, L. H. and Wan, C. C., "Effect of Imperfections on Buckling of Thin Cylinders and Columns under Axial Compression," J. Applied Mechanics, vol. 17, No. 1, p. 73, 1950.
5. Flugge, W., "Die Stabilitaet der Kreiszyllinderschale," Ing. Arch. vol. 3, pp. 463, 1932.
6. Von Kármán, T. and Tsien, H. S., "The Buckling of Thin Cylindrical Shells under Axial Compression," J. Aeronaut. Sci., vol. 8, No. 8, pp. 302-313.
7. Yoshimura, Y., "On the Mechanism of Buckling of a Circular Cylindrical Shell under Axial Compression," NACA TM 1390, 1955.
8. Thielemann, W., "On the Postbuckling Behavior of Thin Cylindrical Shells," NASA, TND-1510, 1962.
9. Uemura, M., "Postbuckling Behavior of a Circular Cylindrical Shell that Buckles Locally under Axial Compression," Department of Aeronautics and Astronautics, Stanford University, S.U.D.A.E.R. No. 56, 1963.
10. Evan-Iwanowski, R. M., Loo, T. C. and Tierney, D. H., "Local Buckling of Shells," Proceedings. 8th Midwestern Mechanics Conference April 3-4, 1963, Case Institute of Technology, Cleveland, Ohio. Developments in Mechanics, vol. 2. pp. 221-251, 1963.
11. Tennyson, R., "A Note on the Classical Buckling Load of Circular Cylindrical Shells under Axial Compression," AIAA Journal 1, pp. 475-476, (1963).
12. Leonard, R., "Comments on, 'A Note on the Classical Buckling Load of Circular Cylindrical Shells under Axial Compression,'" AIAA Journal 1, pp. 2194-2195, (1963).
13. Föppl, L., "A Chsensymmetrisches Ausknicken Zylindrischer Schalen," S. B. Bayr. Akad. Wiss. 1926, pp. 27-40.

14. Flügge, W., "Stresses in Shells," Springer-Verlag, Berlin.
15. Stein, M., "The Influence of Prebuckling Deformations and Stresses on the Buckling of Perfect Cylinders," NASA TR R-190, Feb. 1964.
16. Fischer, G., "Über den Einfluss der gelenkigen Lagerung auf die Stabilität dünnwandiger Kreiszylinderschalen unter Axiallast und Innendruck," Zeitschr Flugwiss. 11, (1963), 111-119.
17. Koiter, W. T., "The Effect of Axisymmetric Imperfections on the Buckling of Cylindrical Shells under Axial Compression," Lockheed Missiles and Space Co., 1963.
18. Hoff, N., "Buckling of Thin Shells," Department of Aeronautical Engineering, Stanford University, S.U.D.A.E.R. No. 114.
19. Nachbar, W. and Hoff, N., "On Edge Buckling of Axially-Compressed, Circular Cylindrical Shells," Department of Aeronautical Engineering, Stanford University, S.U.D.A.E.R. No. 115.
20. Donnell, L. H., "Stability of Thin-Walled Tubes under Torsion," NACA Report No. 479, 1933.
21. Ohira, H., "Local Buckling Theory of Axially Compressed Cylinders," Proceedings of Eleventh Japan National Congress for Applied Mechanics, 1961.
22. Hoff, N. and Redfield, L., "Buckling of Axially Compressed Circular Cylindrical Shells at Stresses Smaller than the Classical Critical Value," Department of Aeronautical Engineering, Stanford University, S.U.D.A.E.R., No. 191.
23. Almroth, B. O., "Influence of Edge Conditions on the Stability of Axially Compressed Cylindrical Shells," NASA C.R.-161, Feb. 1965.
24. Sobel, L. H., "Effects of Boundary Conditions on the Stability of Cylinders Subject to Lateral and Axial Pressures," Lockheed Missiles and Space Co., Sept. 1963.